

Tight lower bounds on the number of faces of the Minkowski sum of convex polytopes via the Cayley trick

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Abstract

Consider a set of r convex d -polytopes P_1, P_2, \dots, P_r , where $d \geq 3$ and $r \geq 2$, and let n_i be the number of vertices of P_i , $1 \leq i \leq r$. It has been shown by Fukuda and Weibel [4] that the number of k -faces of the Minkowski sum, $P_1 + P_2 + \dots + P_r$, is bounded from above by $\Phi_{k+r}(n_1, n_2, \dots, n_r)$, where $\Phi_\ell(n_1, n_2, \dots, n_r) = \sum_{\substack{1 \leq s_i \leq n_i \\ s_1 + \dots + s_r = \ell}} \prod_{i=1}^r \binom{n_i}{s_i}$, $\ell \geq r$. Fukuda and Weibel [4] have also shown that the upper bound mentioned above is tight for $d \geq 4$, $2 \leq r \leq \lfloor \frac{d}{2} \rfloor$, and for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - r$.

In this paper we construct a set of r neighborly d -polytopes P_1, P_2, \dots, P_r , where $d \geq 3$ and $2 \leq r \leq d-1$, for which the upper bound of Fukuda and Weibel is attained for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor - r$. A direct consequence of our result is a tight asymptotic bound on the complexity of the Minkowski sum $P_1 + P_2 + \dots + P_r$, for any fixed dimension d and any $2 \leq r \leq d-1$, when the number of vertices of the polytopes is (asymptotically) the same.

Our approach is based on what is known as the Cayley trick for Minkowski sums: the Minkowski sum, $P_1 + P_2 + \dots + P_r$, is the intersection of the Cayley polytope \mathcal{P} , in $\mathbb{R}^{r-1} \times \mathbb{R}^d$, of the d -polytopes P_1, P_2, \dots, P_r , with an appropriately defined d -flat \overline{W} of $\mathbb{R}^{r-1} \times \mathbb{R}^d$. To prove our bounds, we construct the d -polytopes P_1, P_2, \dots, P_r , where $d \geq 3$ and $2 \leq r \leq d-1$, in such a way so that the number of $(k-1)$ -faces of \mathcal{P} , that intersect the d -flat \overline{W} , is equal to $\Phi_k(n_1, n_2, \dots, n_r)$, for all $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$. The tightness of our bounds then follows from the Cayley trick: the $(k+r-1)$ -faces of the intersection of \mathcal{P} with \overline{W} are in one-to-one correspondence with the k -faces of $P_1 + P_2 + \dots + P_r$, which implies that $f_k(P_1 + P_2 + \dots + P_r) = \Phi_{k+r}(n_1, n_2, \dots, n_r)$, for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor - r$.

Key words: high-dimensional geometry, discrete geometry, combinatorial geometry, Cayley trick, lower bounds, Minkowski sum, convex polytopes

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1 Introduction

Given two sets A and B in \mathbb{R}^d , their Minkowski sum, $A + B$, is defined as the set $\{a + b \mid a \in A, b \in B\}$. Minkowski sums are fundamental structures in both Mathematics and Computer Science. They appear in a wide variety of sub-disciplines, including Combinatorial Geometry, Computational Geometry, Computer Algebra, Computer-Aided Design and Robotics, just to name a few. In recent years, they have found applications in areas such as Game Theory and Computational Biology. It is beyond the scope of this paper to discuss the different uses and applications of Minkowski sums; the interested reader may refer to [15] and [3], and the references therein.

The focus of this work is on Minkowski sums of polytopes, and, in particular, convex polytopes. Tight, or almost tight, asymptotic bounds on the worst-case complexity of the Minkowski sum of two, possibly non-convex, polytopes may be found, e.g., in [1], [14], [2], and [10]. In this paper, we are interested in *exact* bounds on the complexity of the Minkowski sum of two or more polytopes¹. Our aim is to answer a natural and fundamental question: given r d -polytopes, what is the maximum number of k -faces of their Minkowski sum?

For $r \geq 2$ polygons (2-polytopes) P_1, P_2, \dots, P_r , it is known that the number of vertices (or edges) of $P_1 + P_2 + \dots + P_r$ is equal, in the worst-case, to $\sum_{i=1}^r n_i$, where n_i is the number of vertices (or edges) of P_i (see, e.g., [1], [15]). For higher-dimensional polytopes, the first answer to this question was given by Gritzmann and Sturmfels [7]: given r polytopes P_1, P_2, \dots, P_r in \mathbb{R}^d , with a total of n non-parallel edges, the number of l -faces of $P_1 + P_2 + \dots + P_r$ is bounded from above by $2 \binom{n}{l} \sum_{j=1}^{d-1-l} \binom{n-j-1}{j}$. This bound is attained if the polytopes P_i are *zonotopes*, whose generating edges are in general position. Fukuda and Weibel [4] have shown, what they call the *trivial upper bound*: given r d -polytopes P_1, P_2, \dots, P_r , where $d \geq 3$ and $r \geq 2$, we have

$$f_k(P_1 + P_2 + \dots + P_r) \leq \Phi_{k+r}(n_1, n_2, \dots, n_r), \quad (1)$$

where n_i is the number of vertices of P_i , $1 \leq i \leq r$, and

$$\Phi_\ell(n_1, n_2, \dots, n_r) = \sum_{\substack{1 \leq s_i \leq n_i \\ s_1 + \dots + s_r = \ell}} \prod_{i=1}^r \binom{n_i}{s_i}, \quad \ell \geq r, \quad s_i \in \mathbb{N}. \quad (2)$$

In the same paper, Fukuda and Weibel have shown that the trivial upper bound is tight for: (i) $d \geq 4$, $2 \leq r \leq \lfloor \frac{d}{2} \rfloor$ and for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - r$, and (ii) for the number of vertices, $f_0(P_1 + P_2 + \dots + P_r)$, of $P_1 + P_2 + \dots + P_r$, when $d \geq 3$ and $2 \leq r \leq d - 1$. For $r \geq d$, Sanyal [13] has shown that the trivial bound for $f_0(P_1 + P_2 + \dots + P_r)$ cannot be attained, since in this case:

$$f_0(P_1 + P_2 + \dots + P_r) \leq \left(1 - \frac{1}{(d+1)^d}\right) \prod_{i=1}^r n_i < \prod_{i=1}^r n_i.$$

Tight bounds for $f_0(P_1 + P_2 + \dots + P_r)$, where $r \geq d$, have very recently been shown by Weibel [16], namely:

$$f_0(P_1 + P_2 + \dots + P_r) \leq \alpha + \sum_{j=1}^{d-1} (-1)^{d-1-j} \binom{r-1-j}{d-1-j} \sum_{S \in \mathcal{C}_j^r} \left(\prod_{i \in S} f_0(P_i) - \alpha \right),$$

where \mathcal{C}_j^r is the family of subsets of $\{1, 2, \dots, r\}$ of cardinality j , and $\alpha = 2(d - 2 \lfloor \frac{d}{2} \rfloor)$.

¹In the rest of the paper all polytopes are considered to be convex.

Tight bounds for *all* face numbers, i.e., for all $0 \leq k \leq d-1$, are only known for *two* d -polytopes, where $d \geq 3$. Fukuda and Weibel [4] have shown that, given two 3-polytopes P_1 and P_2 in \mathbb{R}^3 , the number of k -faces of $P_1 + P_2$, $0 \leq k \leq 2$, is bounded from above as follows:

$$\begin{aligned} f_0(P_1 + P_2) &\leq n_1 n_2, \\ f_1(P_1 + P_2) &\leq 2n_1 n_2 + n_1 + n_2 - 8, \\ f_2(P_1 + P_2) &\leq n_1 n_2 + n_1 + n_2 - 6, \end{aligned}$$

where n_i is the number of vertices of P_i , $i = 1, 2$. These bounds are tight. Weibel [15] has derived analogous tight expressions in terms of the number of facets m_i of P_i , $i = 1, 2$:

$$\begin{aligned} f_0(P_1 + P_2) &\leq 4m_1 m_2 - 8m_1 - 8m_2 + 16, \\ f_1(P_1 + P_2) &\leq 8m_1 m_2 - 17m_1 - 17m_2 + 40, \\ f_2(P_1 + P_2) &\leq 4m_1 m_2 - 9m_1 - 9m_2 + 26. \end{aligned} \tag{3}$$

Weibel's expression for $f_2(P_1 + P_2)$ (cf. (3)) has been generalized to the number of facets of the Minkowski sum of any number of 3-polytopes by Fogel, Halperin and Weibel [2]; they have shown that, for $r \geq 2$, the following tight bound holds:

$$f_2(P_1 + P_2 + \cdots + P_r) \leq \sum_{1 \leq i < j \leq r} (2m_i - 5)(2m_j - 5) + \sum_{i=1}^r m_i + \binom{r}{2},$$

where $m_i = f_2(P_i)$, $1 \leq i \leq r$. Regarding d -polytopes, where $d \geq 4$, Karavelas and Tzanaki [11, 12], have shown that, for $1 \leq k \leq d$:

$$f_{k-1}(P_1 + P_2) \leq f_k(C_{d+1}(n_1 + n_2)) - \sum_{i=0}^{\lfloor \frac{d+1}{2} \rfloor} \binom{d+1-i}{k+1-i} \left(\binom{n_1-d-2+i}{i} + \binom{n_2-d-2+i}{i} \right), \tag{4}$$

where $n_i = f_0(P_i)$, $i = 1, 2$, and $C_d(n)$ stands for the cyclic d -polytope with n vertices. The bounds in (4) have been shown to be tight for any $d \geq 3$ and for all $1 \leq k \leq d$, and, clearly, match the corresponding bounds for 2- and 3-polytopes (cf. rel. (3)), as well as the expressions in (1) for $r = 2$ and for all $0 \leq k \leq \lfloor \frac{d+1}{2} \rfloor - 2$. Notice that the tightness of relations (4), implies that the trivial upper bounds in (1) are also tight for $d \geq 3$, $r = 2$ and $k \leq \lfloor \frac{d+1}{2} \rfloor - 2$ (as opposed to $d \geq 4$, $r = 2$ and $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - 2$).

In this paper, we show that the trivial upper bound (1) is attained for a wider range of d , r and k than those proved by Fukuda and Weibel [4]. More precisely, we prove that for any $d \geq 3$, $2 \leq r \leq d-1$ and for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor - r$, there exist r neighborly d -polytopes P_1, P_2, \dots, P_r , for which the number of k -faces of their Minkowski sum attains the trivial upper bound. Our approach is based on what is known as the *Cayley trick* for Minkowski sums. Let \mathcal{V}_i be the vertex set of P_i , $1 \leq i \leq r$, and let $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_r$ be an affine basis of \mathbb{R}^{r-1} . The *Cayley embedding* $\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r)$, in $\mathbb{R}^{r-1} \times \mathbb{R}^d$, of the vertex sets $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r$, is defined as $\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r) = \cup_{i=1}^r \{(\mathbf{b}_i, \mathbf{v}) \mid \mathbf{v} \in \mathcal{V}_i\}$. The Minkowski sum, $P_1 + P_2 + \cdots + P_r$, can then be viewed as the intersection of the *Cayley polytope* $\mathcal{P} = \text{conv}(\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r))$ with an appropriately defined d -flat $\overline{\mathcal{W}}$ of $\mathbb{R}^{r-1} \times \mathbb{R}^d$. We exploit this observation in two steps. We first construct a set of r $(d-r+1)$ -polytopes Q_1, Q_2, \dots, Q_r , with n_1, n_2, \dots, n_r vertices, respectively, embedded in appropriate subspaces of \mathbb{R}^d . The polytopes Q_1, Q_2, \dots, Q_r are constructed in such a way so that the number of $(k-1)$ -faces of the set \mathcal{W}_Q is equal to $\Phi_k(n_1, n_2, \dots, n_r)$ for all $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$, where \mathcal{W}_Q is the set of faces of the Cayley polytope \mathcal{Q} that have non-empty intersection with $\overline{\mathcal{W}}$.

We then perturb, via a single perturbation parameter ζ , the vertices of the Q_i 's to get a set of r full-dimensional (i.e., d -dimensional) neighborly polytopes P_1, P_2, \dots, P_r ; we next consider the Cayley polytope \mathcal{P} of the P_i 's, and show that is possible to choose a small positive value for ζ , so that the number of $(k-1)$ -faces of $\mathcal{W}_{\mathcal{P}}$ is equal to $\Phi_k(n_1, n_2, \dots, n_r)$ for all $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$, where $\mathcal{W}_{\mathcal{P}}$ is the set of faces of \mathcal{P} with non-empty intersection with \overline{W} . Our tight lower bound then follows from the fact that the $(k-1)$ -faces of $\mathcal{W}_{\mathcal{P}}$ are in one-to-one correspondence with the $(k-r)$ -faces of $P_1 + P_2 + \dots + P_r$, for all $r \leq k \leq d+r-1$.

Beyond extending, with respect to d , r and k , the range of tightness of the trivial upper bound in (1), our lower bound construction possesses some additional interesting characteristics:

1. It gives, as a special case, Fukuda and Weibel's tight bound on the number of vertices of the Minkowski sum of r d -polytopes for $d \geq 3$ and $2 \leq r \leq d-1$ (cf. [4, Theorem 2]).
2. It constitutes a generalization of the lower bound construction used in [11, 12] to prove the tightness of relation (4) for $k = \lfloor \frac{d+1}{2} \rfloor - 2$, and any odd $d \geq 3$.
3. For any fixed dimension $d \geq 3$ and any $2 \leq r \leq d-1$, it implies a tight bound on the complexity of the Minkowski sum of r d -polytopes, when the polytopes have (asymptotically) the same number of vertices. Notice that the complexity of the Minkowski sum of r d -polytopes is bounded from above by the complexity of their Cayley polytope, which is in $O(n^{\lfloor \frac{d+r-1}{2} \rfloor})$ when we assume that the polytopes have $\Theta(n)$ vertices. On the other hand, if $n_i = \Theta(n)$, $1 \leq i \leq r$, our construction yields $f_k(P_1 + P_2 + \dots + P_r) = \Theta(n^{k+r})$, $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor - r$, and, in particular, $f_{\lfloor \frac{d+r-1}{2} \rfloor - r}(P_1 + P_2 + \dots + P_r) = \Theta(n^{\lfloor \frac{d+r-1}{2} \rfloor})$.
4. It gives the maximum possible ranges of d , r and k for which the k -faces of the Minkowski sum of r d -polytopes is equal to $\Phi_{k+r}(n_1, n_2, \dots, n_r)$. If, on the contrary, the trivial upper bound was attained for some $k > \lfloor \frac{d+r-1}{2} \rfloor - r$, we would have that the complexity of the Minkowski sum of r n -vertex d -polytopes would be in $\Omega(n^{\lfloor \frac{d+r-1}{2} \rfloor + 1})$. For $2 \leq r \leq d-1$, this directly contradicts with the discussion in the previous item, while for $r \geq d \geq 3$, it is known that the complexity of the Minkowski sum of r n -vertex d -polytopes is in $O(r^{d-1}n^{d-1})$ (cf. [16]).

Finally, we believe that our result is *optimal* in the sense that, for any $d \geq 3$ and any $2 \leq r \leq d-1$, the Minkowski sum of the r d -polytopes P_1, P_2, \dots, P_r that we construct, has the maximum possible number of k -faces for all $0 \leq k \leq d-1$. This has been proved to be true for the case of two d -polytopes and for any odd $d \geq 3$ (cf. [11, 12]), while it is straightforward to show that it also holds true for the case of two d -polytopes and any even $d \geq 4$.

The remaining sections of this paper are as follows. In Section 2 we give some definitions, describe the Cayley trick, and discuss its consequences that are relevant to our results. In Section 3 we present the construction that establishes the tightness of the trivial upper bound for $d \geq 3$, $2 \leq r \leq d-1$ and $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor - r$. We conclude with Section 4, where we discuss our results and state directions for future research.

2 Preliminaries

A *convex polytope*, or simply *polytope*, P in \mathbb{R}^d is the convex hull of a finite set of points V in \mathbb{R}^d , called the *vertex set* of P . A *face* of P is the intersection of P with a hyperplane for which the polytope is contained in one of the two closed halfspaces delimited by the hyperplane. The dimension of a face of P is the dimension of its affine hull. A k -face of P is a k -dimensional face of P . We consider the polytope itself as a trivial d -dimensional face; all the other faces are called

proper faces. We use the term *d-polytope* to refer to a polytope the trivial face of which is d -dimensional. For a d -polytope P , the 0-faces of P are its *vertices*, while the $(d-1)$ -faces are called *facets*. For $0 \leq k \leq d$ we denote by $f_k(P)$ the number of k -faces of P . Note that every k -face F of P is also a k -polytope whose faces are all the faces of P contained in F . Finally, a d -polytope P is called *k-neighborly*, if every subset of its vertices of size at most k defines a face of P . The maximum possible level of *neighborliness* of a d -polytope is $\lfloor \frac{d}{2} \rfloor$. A $\lfloor \frac{d}{2} \rfloor$ -neighborly d -polytope is simply referred to as *neighborly*.

The Cayley trick. Let P_1, P_2, \dots, P_r be r d -polytopes with vertex sets $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r$, respectively. Let $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_r$ be an affine basis of \mathbb{R}^{r-1} , and call $\mu_i : \mathbb{R}^d \rightarrow \mathbb{R}^{r-1} \times \mathbb{R}^d$, the affine inclusion given by $\mu_i(\mathbf{x}) = (\mathbf{b}_i, \mathbf{x})$. The *Cayley embedding* $\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r)$ of the point sets $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r$ is defined as $\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r) = \cup_{i=1}^r \mu_i(\mathcal{V}_i)$. The polytope corresponding to the convex hull $\text{conv}(\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r))$ of the Cayley embedding $\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r)$ of $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r$ is typically referred to as the *Cayley polytope* of P_1, P_2, \dots, P_r .

Let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a *weight vector*, i.e., $\lambda_i > 0$, $1 \leq i \leq r$, and $\sum_{i=1}^r \lambda_i = 1$. The following lemma, known as *the Cayley trick for Minkowski sums*, relates the $\boldsymbol{\lambda}$ -weighted Minkowski sum of the polytopes P_1, P_2, \dots, P_r with the Cayley polytope of these polytopes.

Lemma 1 ([9, Lemma 3.2]). *Let P_1, P_2, \dots, P_r be r d -polytopes with vertex sets $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r \subset \mathbb{R}^d$. Moreover, let $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a weight vector, and $W(\boldsymbol{\lambda}) := \{\lambda_1 \mathbf{b}_1 + \dots + \lambda_r \mathbf{b}_r\} \times \mathbb{R}^d \subset \mathbb{R}^{r-1} \times \mathbb{R}^d$. Then, the $\boldsymbol{\lambda}$ -weighted Minkowski sum $\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_r P_r \subset \mathbb{R}^d$ has the following representation as a section of the Cayley embedding $\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r)$ in $\mathbb{R}^{r-1} \times \mathbb{R}^d$:*

$$\begin{aligned} \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_r P_r &\cong \mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r) \wedge W(\boldsymbol{\lambda}) \\ &:= \left\{ \text{conv}\{(\mathbf{b}_1, \mathbf{v}_1), (\mathbf{b}_2, \mathbf{v}_2), \dots, (\mathbf{b}_r, \mathbf{v}_r)\} \cap W(\boldsymbol{\lambda}) : \right. \\ &\quad \left. (\mathbf{b}_1, \mathbf{v}_1), (\mathbf{b}_2, \mathbf{v}_2), \dots, (\mathbf{b}_r, \mathbf{v}_r) \in \mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r) \right\}. \end{aligned}$$

Moreover, F is a facet of $\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_r P_r$ if and only if it is of the form $F = F' \wedge W(\boldsymbol{\lambda})$ for a facet F' of $\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r)$ containing at least one point $(\mathbf{b}_i, \mathbf{v}_i)$ for all $1 \leq i \leq r$.

As described in [9, Corollary 3.7], the $\boldsymbol{\lambda}$ -weighted Minkowski sums of r polytopes P_1, P_2, \dots, P_r have isomorphic posets of subdivisions for different values of $\boldsymbol{\lambda}$. This implies that, for any weight vector $\boldsymbol{\lambda}$, the $\boldsymbol{\lambda}$ -weighted Minkowski sum is equivalent to the $\bar{\boldsymbol{\lambda}}$ -weighted Minkowski sum, where $\bar{\boldsymbol{\lambda}}$ is the averaging weight vector: $\bar{\boldsymbol{\lambda}} = (\frac{1}{r}, \frac{1}{r}, \dots, \frac{1}{r})$. On the other hand, the $\bar{\boldsymbol{\lambda}}$ -weighted Minkowski sum is nothing but a scaled version of the unweighted Minkowski sum $P_1 + P_2 + \dots + P_r$, i.e., for any weight vector $\boldsymbol{\lambda}$, the $\boldsymbol{\lambda}$ -weighted Minkowski sum is combinatorially equivalent to the unweighted Minkowski sum. In that respect, in the rest of the paper we only consider the (unweighted) Minkowski sum of P_1, P_2, \dots, P_r , while our results carry over to $\boldsymbol{\lambda}$ -weighted Minkowski sums, for any weight vector $\boldsymbol{\lambda}$.

Let \mathcal{P} be the Cayley polytope of P_1, P_2, \dots, P_r , and call $\mathcal{W}_{\mathcal{P}}$ the set of faces of \mathcal{P} that have non-empty intersection with the d -flat $\overline{W} = W(\bar{\boldsymbol{\lambda}})$. A direct consequence of Lemma 1 is a bijection between the $(k-1)$ -faces of $\mathcal{W}_{\mathcal{P}}$ and the $(k-r)$ -faces of $P_1 + P_2 + \dots + P_r$, for $r \leq k \leq d+r-1$. This further implies that

$$f_{k-1}(\mathcal{W}_{\mathcal{P}}) = f_{k-r}(P_1 + P_2 + \dots + P_r), \quad r \leq k \leq d+r-1. \quad (5)$$

3 Lower bound construction

Given a set \mathcal{S} and a partition \mathcal{S} of \mathcal{S} into r subsets S_1, S_2, \dots, S_r , we say that S is a *\mathcal{S} -spanning subset* of \mathcal{S} if $S \cap S_i \neq \emptyset$ for all $1 \leq i \leq r$. Assuming that n_i is the cardinality of S_i , the number of

S-spanning subsets of \mathcal{S} of size $k \geq r$ is equal to $\Phi_k(n_1, n_2, \dots, n_r)$ (cf. (2)).

In what follows we assume that $d \geq 3$ and $2 \leq r \leq d-1$. We denote by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{r-1}$ the standard basis of \mathbb{R}^{r-1} , while we use \mathbf{e}_0 to denote the zero vector in \mathbb{R}^{r-1} . Notice that the vectors $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{r-1}$ form an affine basis of \mathbb{R}^{r-1} . Consider a set of r d -polytopes P_1, P_2, \dots, P_r , where $2 \leq r \leq d-1$. Let \mathcal{V}_i be the vertex set of P_i , $1 \leq i \leq r$, let $\mathcal{V} = \cup_{i=1}^r \mathcal{V}_i$, and call \mathcal{V} the partition of \mathcal{V} into its subsets $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r$. Let \mathcal{P} be the Cayley polytope of P_1, P_2, \dots, P_r , where, in order to perform the Cayley embedding, we have chosen $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{r-1}$ as the affine basis of \mathbb{R}^{r-1} . Let \overline{W} denote the d -flat

$$\overline{W} = \{\frac{1}{r}\mathbf{e}_0 + \frac{1}{r}\mathbf{e}_1 + \dots + \frac{1}{r}\mathbf{e}_{r-1}\} \times \mathbb{R}^d \quad (6)$$

of $\mathbb{R}^{r-1} \times \mathbb{R}^d$. Call $\mathcal{W}_{\mathcal{P}}$ the set of faces of \mathcal{P} that have non-empty intersection with \overline{W} . As described in the previous section, the intersection of \mathcal{P} with \overline{W} is combinatorially equivalent to the Minkowski sum $P_1 + P_2 + \dots + P_r$, and, in particular, the k -faces of $P_1 + P_2 + \dots + P_r$ are in one-to-one correspondence with the $(k+r-1)$ -faces of $\mathcal{W}_{\mathcal{P}}$, $0 \leq k \leq d-1$. In fact, the faces of $\mathcal{W}_{\mathcal{P}}$ are precisely the faces of \mathcal{P} whose vertex sets are \mathcal{V} -spanning subsets of \mathcal{V} . In view of relation (5), maximizing the value of $f_{k-r}(P_1 + P_2 + \dots + P_r)$ is equivalent to maximizing the value of $f_{k-1}(\mathcal{W}_{\mathcal{P}})$, where $2 \leq r \leq d-1$ and $r \leq k \leq d+r-1$. In this section we exploit this observation, so as to construct a set of r d -polytopes P_1, P_2, \dots, P_r , with n_1, n_2, \dots, n_r vertices, respectively, for which the number of $(k-1)$ -faces of $\mathcal{W}_{\mathcal{P}}$ attains its maximal possible value $\Phi_k(n_1, n_2, \dots, n_r)$, for all $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$.

Before getting into the technical details, we first outline our approach. We start by considering the $(d-r+1)$ -dimensional moment curve, which we embed in r distinct subspaces of \mathbb{R}^d . We consider the r copies of the $(d-r+1)$ -dimensional moment curve as different curves, and we perturb them appropriately, so that they become d -dimensional moment-like curves. The perturbation is controlled via a non-negative parameter ζ , which will be chosen appropriately. We then choose points on these r moment-like curves, all parameterized by a positive parameter τ , which will again be chosen appropriately. We call P_1, P_2, \dots, P_r the r d -polytopes we get by considering the points on each moment-like curve, \mathcal{P} the Cayley polytope of P_1, P_2, \dots, P_r , and $\mathcal{W}_{\mathcal{P}}$ the set of faces of \mathcal{P} that have non-empty intersection with \overline{W} . For these polytopes we show that the number of $(k-1)$ -faces of $\mathcal{W}_{\mathcal{P}}$, where $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$, becomes equal to $\Phi_k(n_1, n_2, \dots, n_r)$ for small enough positive values of ζ and τ .

At a more technical level, the proof that $f_{k-1}(\mathcal{W}_{\mathcal{P}}) = \Phi_k(n_1, n_2, \dots, n_r)$, for all $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$, is performed in two steps. We first consider the cyclic $(d-r+1)$ -polytopes Q_1, Q_2, \dots, Q_r , embedded in appropriate subspaces of \mathbb{R}^d . The Q_i 's are the *unperturbed*, with respect to ζ , versions of the d -polytopes P_1, P_2, \dots, P_r (i.e., the polytope Q_i is the polytope we get from P_i , when we set ζ equal to zero). We consider the Cayley polytope \mathcal{Q} of Q_1, Q_2, \dots, Q_r , seen as polytopes in \mathbb{R}^d , and we focus on the set $\mathcal{W}_{\mathcal{Q}}$ of faces of \mathcal{Q} , that are the faces of \mathcal{Q} intersected by \overline{W} . Noticing that the polytopes Q_1, Q_2, \dots, Q_r are parameterized by the parameter τ , we show that there exists a sufficiently small positive value for τ , for which the number of $(k-1)$ -faces of $\mathcal{W}_{\mathcal{Q}}$ is equal to $\Phi_k(n_1, n_2, \dots, n_r)$. Having chosen the appropriate value for τ , which we denote by τ^* , we then consider the polytopes P_1, P_2, \dots, P_r (with τ set to τ^*), and show that for sufficiently small ζ , $f_{k-1}(\mathcal{W}_{\mathcal{P}})$ is equal to $\Phi_k(n_1, n_2, \dots, n_r)$.

We start off with a technical lemma and sketch its proof. The detailed proof may be found in Section A of the Appendix.

Lemma 2. *Let $\kappa_1, \dots, \kappa_n$ be $n \geq 2$ integers such that $\kappa_i \geq 2$, $1 \leq i \leq n$, and let $K = \sum_{i=1}^n \kappa_i$. Let $x_{i,j}$ be real numbers such that $0 < x_{i,1} < x_{i,2} < \dots < x_{i,\kappa_i}$, $1 \leq i \leq n$. Let β_i , $1 \leq i \leq n$ be*

non-negative integers such that $\beta_1 > \beta_2 > \dots > \beta_n \geq 0$. Finally, let τ be a positive real parameter, and define $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ to be the determinant:

$$(-1)^N \begin{vmatrix} \overbrace{\begin{matrix} 1 & \dots & 1 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{matrix}}^{\kappa_1} & \overbrace{\begin{matrix} 0 & \dots & 0 \\ 1 & \dots & 1 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{matrix}}^{\kappa_2} & \overbrace{\begin{matrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 1 & \dots & 1 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{matrix}}^{\kappa_3} & \dots & \overbrace{\begin{matrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{matrix}}^{\kappa_n} \\ x_{1,1}\tau^{\beta_1} \dots x_{1,\kappa_1}\tau^{\beta_1} & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & x_{2,1}\tau^{\beta_2} & \dots & x_{2,\kappa_2}\tau^{\beta_2} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & x_{3,1}\tau^{\beta_3} & \dots & x_{3,\kappa_3}\tau^{\beta_3} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & x_{n,1}\tau^{\beta_n} & \dots & x_{n,\kappa_n}\tau^{\beta_n} \\ x_{1,1}^2\tau^{2\beta_1} \dots x_{1,\kappa_1}^2\tau^{2\beta_1} & x_{2,1}^2\tau^{2\beta_2} & \dots & x_{2,\kappa_2}^2\tau^{2\beta_2} & x_{3,1}^2\tau^{2\beta_3} & \dots & x_{3,\kappa_3}^2\tau^{2\beta_3} & \dots & x_{n,1}^2\tau^{2\beta_n} & \dots & x_{n,\kappa_n}^2\tau^{2\beta_n} \\ x_{1,1}^3\tau^{3\beta_1} \dots x_{1,\kappa_1}^3\tau^{3\beta_1} & x_{2,1}^3\tau^{3\beta_2} & \dots & x_{2,\kappa_2}^3\tau^{3\beta_2} & x_{3,1}^3\tau^{3\beta_3} & \dots & x_{3,\kappa_3}^3\tau^{3\beta_3} & \dots & x_{n,1}^3\tau^{3\beta_n} & \dots & x_{n,\kappa_n}^3\tau^{3\beta_n} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ x_{1,1}^m\tau^{m\beta_1} \dots x_{1,\kappa_1}^m\tau^{m\beta_1} & x_{2,1}^m\tau^{m\beta_2} & \dots & x_{2,\kappa_2}^m\tau^{m\beta_2} & x_{3,1}^m\tau^{m\beta_3} & \dots & x_{3,\kappa_3}^m\tau^{m\beta_3} & \dots & x_{n,1}^m\tau^{m\beta_n} & \dots & x_{n,\kappa_n}^m\tau^{m\beta_n} \end{vmatrix} \begin{matrix} \left. \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \right\} n \\ \left. \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \right\} n \\ \left. \begin{matrix} \vdots \\ \vdots \\ \vdots \end{matrix} \right\} K-2n \end{matrix}$$

where $N = \frac{n(n-1)}{2}$ and $m = K - 2n + 1$. Then, there exists some $\tau_0 > 0$, such that for all $\tau \in (0, \tau_0)$, the determinant $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ is strictly positive.

Sketch of proof. Let $K_i = \sum_{j=1}^i \kappa_j$, and let $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_n)$, where \mathbf{c}_i is the column vector corresponding to the columns $K_{i-1} + 1$ to K_i of $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$, $1 \leq i \leq n$ (by convention $K_0 = 0$). Let $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n)$ denote a row vector of $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$, where $\mathbf{r}_i \in \mathbb{N}^{\kappa_i}$. We use Laplace's Expansion Theorem, to expand $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ along its first κ_1 columns, then along its next κ_2 columns, etc. This produces an expansion for $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$, consisting of $\prod_{i=1}^n \binom{K-K_{i-1}}{\kappa_i}$ terms, where each term corresponds to a different choice for \mathbf{r} . More precisely, each term is, up to a sign, the product of n minors $M_i(\tau)$ of $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ of size $\kappa_i \times \kappa_i$, $1 \leq i \leq n$, where the columns (resp., rows) of $M_i(\tau)$ are the columns (resp., rows) in \mathbf{c}_i (resp., \mathbf{r}_i). Among these terms, there are $\prod_{i=1}^n \binom{K-2n-K_{i-1}}{\kappa_i-2}$ non-vanishing terms, each of which is of the form $(-1)^{\sigma(\mathbf{r}, \mathbf{c})+N} \tau^{\theta(\mathbf{r}, \mathbf{c})} \prod_{i=1}^n D_i$, where D_i is a generalized Vandermonde determinant. We proceed by identifying the unique, with respect to \mathbf{r} , term in the expansion, for which $\theta(\mathbf{r}, \mathbf{c})$ is minimal. Denoting by $\boldsymbol{\rho}$ the row vector for which this minimal value is attained, we show that $\sigma(\boldsymbol{\rho}, \mathbf{c}) + N$ is even. We, thus, get $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau) = \tau^{\theta(\boldsymbol{\rho}, \mathbf{c})} \prod_{i=1}^n D_i + O(\tau^{\theta(\boldsymbol{\rho}, \mathbf{c})+1})$. Our result then follows by taking the limit $\lim_{\tau \rightarrow 0+} \frac{\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)}{\tau^{\theta(\boldsymbol{\rho}, \mathbf{c})}}$, and by noticing that the determinants D_i , $1 \leq i \leq n$, are strictly positive. \square

Let $\boldsymbol{\gamma}(t)$, $t > 0$, be the $(d-r+1)$ -dimensional moment curve, i.e., $\boldsymbol{\gamma}(t) = (t, t^2, \dots, t^{d-r+1})$. We are going to call $\boldsymbol{\gamma}_i(t)$, $1 \leq i \leq r$, the curve $\boldsymbol{\gamma}(t)$ embedded in the $(d-r+1)$ -flat F_i of \mathbb{R}^d , where

$$F_i = \{x_j = 0 \mid 1 \leq j \leq r \text{ and } j \neq i\}, \quad 1 \leq i \leq r, \quad (7)$$

such that the first coordinate of $\boldsymbol{\gamma}(t)$ becomes the i -coordinate of $\boldsymbol{\gamma}_i(t)$, whereas, for $2 \leq j \leq d-r+1$, the j -th coordinate of $\boldsymbol{\gamma}(t)$ becomes the $(j+r-1)$ -coordinate of $\boldsymbol{\gamma}_i(t)$. In other words:

$$\begin{aligned} \boldsymbol{\gamma}_1(t) &= (\overbrace{t, 0, 0, 0, \dots, 0, 0}^r, \overbrace{t^2, \dots, t^{d-r+1}}^{d-r}), \\ \boldsymbol{\gamma}_2(t) &= (0, t, 0, 0, \dots, 0, 0, t^2, \dots, t^{d-r+1}), \\ \boldsymbol{\gamma}_3(t) &= (0, 0, t, 0, \dots, 0, 0, t^2, \dots, t^{d-r+1}), \\ &\vdots \\ \boldsymbol{\gamma}_r(t) &= (0, 0, 0, 0, \dots, 0, t, t^2, \dots, t^{d-r+1}). \end{aligned}$$

We next *perturb* the vanishing coordinates of $\gamma_i(t)$, $1 \leq i \leq r$, to get the d -dimensional curve $\gamma_i(t; \zeta)$ as follows: the first (from the left) vanishing coordinate of $\gamma_i(t)$ becomes ζt^{d-r+2} , the second vanishing coordinate of $\gamma_i(t)$ becomes ζt^{d-r+3} , etc., and, finally, the last vanishing coordinate of $\gamma_i(t)$ becomes ζt^d :

$$\begin{aligned}\gamma_1(t; \zeta) &= (\overbrace{t, \zeta t^{d-r+2}, \zeta t^{d-r+3}, \zeta t^{d-r+4}, \dots, \zeta t^{d-1}, \zeta t^d}^r, \overbrace{t^2, \dots, t^{d-r+1}}^{d-r}), \\ \gamma_2(t; \zeta) &= (\zeta t^{d-r+2}, t, \zeta t^{d-r+3}, \zeta t^{d-r+4}, \dots, \zeta t^{d-1}, \zeta t^d, t^2, \dots, t^{d-r+1}), \\ \gamma_3(t; \zeta) &= (\zeta t^{d-r+2}, \zeta t^{d-r+3}, t, \zeta t^{d-r+4}, \dots, \zeta t^{d-1}, \zeta t^d, t^2, \dots, t^{d-r+1}), \\ &\vdots \\ \gamma_r(t; \zeta) &= (\zeta t^{d-r+2}, \zeta t^{d-r+3}, \zeta t^{d-r+4}, \dots, \zeta t^{d-1}, \zeta t^d, t, t^2, \dots, t^{d-r+1}),\end{aligned}$$

where $\zeta \geq 0$ is the perturbation parameter (clearly, for $\zeta = 0$, $\gamma_i(t; \zeta)$ reduces to $\gamma_i(t)$). Denote by $\beta_i(t)$ (resp., $\beta_i(t; \zeta)$) the Cayley embedding of $\gamma_i(t)$ (resp., $\gamma_i(t; \zeta)$), i.e.,:

$$\begin{aligned}\beta_i(t) &= \mu_i(\gamma_i(t)) = (e_{i-1}, \gamma_i(t)), \\ \beta_i(t; \zeta) &= \mu_i(\gamma_i(t; \zeta)) = (e_{i-1}, \gamma_i(t; \zeta)),\end{aligned} \quad t > 0, \quad 1 \leq i \leq r.$$

For each i with $1 \leq i \leq r$, choose n_i real numbers $\alpha_{i,j}$, $j = 1, \dots, n_i$, such that $0 < \alpha_{i,1} < \alpha_{i,2} < \dots < \alpha_{i,n_i}$. Let τ be a strictly positive parameter that is determined below, and choose r non-negative integers $\nu_1, \nu_2, \dots, \nu_r$, such that $\nu_1 > \nu_2 > \dots > \nu_r = 0$. Let \mathcal{U}_i , $1 \leq i \leq r$, be the $(d-r+1)$ -dimensional point sets:

$$\mathcal{U}_i = \{\gamma_i(\alpha_{i,1}\tau^{\nu_i}), \gamma_i(\alpha_{i,2}\tau^{\nu_i}), \dots, \gamma_i(\alpha_{i,n_i}\tau^{\nu_i})\}.$$

Since \mathcal{U}_i consists of points on the $(d-r+1)$ -dimensional moment curve $\gamma(t)$, embedded in the $(d-r+1)$ -flat F_i of \mathbb{R}^d , the $(d-r+1)$ -polytope $Q_i = \text{conv}(\mathcal{U}_i)$ is the cyclic $(d-r+1)$ -polytope embedded in F_i (cf. (7)).

Let $\mathcal{U}'_i = \mu_i(\mathcal{U}_i)$, $1 \leq i \leq r$, $\mathcal{U} = \cup_{i=1}^r \mathcal{U}_i$, $\mathcal{U}' = \cup_{i=1}^r \mathcal{U}'_i$, and denote by \mathcal{U} (resp., \mathcal{U}') the partition of \mathcal{U} (resp., \mathcal{U}') into its subsets $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_r$ (resp., $\mathcal{U}'_1, \mathcal{U}'_2, \dots, \mathcal{U}'_r$). Let $\mathcal{Q} = \text{conv}(\mathcal{U}')$ be the Cayley polytope of Q_1, Q_2, \dots, Q_r , and let $\mathcal{W}_{\mathcal{Q}}$ be the set of faces of \mathcal{Q} with non-empty intersection with $\overline{\mathcal{W}}$, i.e., $\mathcal{W}_{\mathcal{Q}}$ consists of all the faces of \mathcal{Q} , the vertex set of which is a \mathcal{U}' -spanning subset of \mathcal{U}' . The following lemma establishes the first step towards our construction.

Lemma 3. *There exists a sufficiently small positive value τ^* for τ , such that*

$$f_{k-1}(\mathcal{W}_{\mathcal{Q}}) = \Phi_k(n_1, n_2, \dots, n_r), \quad r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor.$$

Proof. To simplify the notation used in the proof, we identify \mathcal{U}'_i , \mathcal{U}' and \mathcal{U}' with their pre-images under the Cayley embedding, namely, \mathcal{U}_i , \mathcal{U} and \mathcal{U} , respectively.

Let $\alpha_{i,j}^\epsilon = \alpha_{i,j} + \epsilon$, $t_{i,j}^\epsilon = \alpha_{i,j}\tau^{\nu_i}$, $t_{i,j}^\epsilon = \alpha_{i,j}^\epsilon\tau^{\nu_i}$, where $1 \leq j \leq n_i$, $1 \leq i \leq r$, and $\epsilon > 0$. The value of ϵ is chosen such that $\alpha_{i,j}^\epsilon < \alpha_{i,j+1}$, for all $1 \leq j < n_i$, and for all $1 \leq i \leq r$. Finally, let M be a positive real number such that $M > \alpha_{r,n_r}^\epsilon = \alpha_{r,n_r}^\epsilon\tau^{\nu_r} = t_{r,n_r}^\epsilon$ (recall that $\nu_r = 0$).

Choose a \mathcal{U} -spanning subset U of \mathcal{U} of size k , and denote by k_i the cardinality of $U_i = U \cap \mathcal{U}_i$; clearly, $\sum_{i=1}^r k_i = k$. Let $\beta_i(t_{i,j_{i,1}}), \beta_i(t_{i,j_{i,2}}), \dots, \beta_i(t_{i,j_{i,k_i}})$, be the vertices in U_i , where $j_{i,1} < j_{i,2} < \dots < j_{i,k_i}$ and $1 \leq i \leq r$. Let $\mathbf{x} = (x_1, x_2, \dots, x_{d+r-1})$ and define the $(d+r) \times (d+r)$ determinant

$H_U(\mathbf{x})$ as follows²:

$$H_U(\mathbf{x}) = (-1)^R \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ \mathbf{x} \beta_1(t_{1,j_{1,1}}) & \beta_1(t_{1,j_{1,1}}^\epsilon) & \cdots & \beta_1(t_{1,j_{1,k_1}}) & \beta_1(t_{1,j_{1,k_1}}^\epsilon) & \beta_2(t_{2,j_{2,1}}) & \beta_2(t_{2,j_{2,1}}^\epsilon) & \cdots & \beta_2(t_{2,j_{2,k_2}}) & \beta_2(t_{2,j_{2,k_2}}^\epsilon) \\ \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ \cdots & \beta_{r-1}(t_{r-1,j_{r-1,1}}) & \beta_{r-1}(t_{r-1,j_{r-1,1}}^\epsilon) & \cdots & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}) & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}^\epsilon) & \cdots & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}) & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}^\epsilon) \\ \beta_r(t_{r,j_{r,1}}) & \beta_r(t_{r,j_{r,1}}^\epsilon) & \cdots & \beta_r(t_{r,j_{r,k_r}}) & \beta_r(t_{r,j_{r,k_r}}^\epsilon) & \beta_r(M) & \beta_r(2M) & \cdots & \beta_r((d+r-1-2k)M) \end{vmatrix}, \quad (8)$$

where $R = \frac{r(r-1)}{2}$. We can alternatively describe $H_U(\mathbf{x})$ as follows:

- (i) The first column of $H_U(\mathbf{x})$ is $\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}$.
- (ii) For i ranging from 1 to r , and for λ ranging from 1 to k_i , the next k_i pairs of columns of $H_U(\mathbf{x})$ are $\begin{pmatrix} 1 \\ \beta_i(t_{i,j_{i,\lambda}}) \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \beta_i(t_{i,j_{i,\lambda}}^\epsilon) \end{pmatrix}$.
- (iii) For λ ranging from 1 to $d+r-1-2k$, the last $d+r-1-2k$ columns of $H_U(\mathbf{x})$ are $\begin{pmatrix} 1 \\ \beta_r(t_{\lambda M}) \end{pmatrix}$. Notice that if $k = \lfloor \frac{d+r-1}{2} \rfloor$ and $d+r-1$ is even, this last category of columns of $H_U(\mathbf{x})$ does not exist.

The equation $H_U(\mathbf{x}) = 0$ is the equation of a hyperplane in \mathbb{R}^{d+r-1} that passes through the points in U . We are going to show that, for any choice of U , and for all vertices \mathbf{u} in $\mathcal{U} \setminus U$, we have $H_U(\mathbf{u}) > 0$ for sufficiently small τ .

Suppose we have some vertex \mathbf{u} of \mathcal{U} such that $\mathbf{u} \in \mathcal{U} \setminus U$. Then, $\mathbf{u} = \beta_i(t_{i,\lambda})$, $t_{i,\lambda} = \alpha_{i,\lambda} \tau^{\nu_i}$, for some $\lambda \notin \{j_{i,1}, j_{i,2}, \dots, j_{i,k_i}\}$. Then $H_U(\mathbf{u})$ becomes:

$$H_U(\mathbf{u}) = (-1)^R \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ \beta_i(t_{i,\lambda}) & \beta_1(t_{1,j_{1,1}}) & \beta_1(t_{1,j_{1,1}}^\epsilon) & \cdots & \beta_1(t_{1,j_{1,k_1}}) & \beta_1(t_{1,j_{1,k_1}}^\epsilon) & \beta_2(t_{2,j_{2,1}}) & \beta_2(t_{2,j_{2,1}}^\epsilon) & \cdots & \beta_2(t_{2,j_{2,k_2}}) & \beta_2(t_{2,j_{2,k_2}}^\epsilon) \\ \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ \cdots & \beta_{r-1}(t_{r-1,j_{r-1,1}}) & \beta_{r-1}(t_{r-1,j_{r-1,1}}^\epsilon) & \cdots & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}) & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}^\epsilon) & \cdots & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}) & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}^\epsilon) \\ \beta_r(t_{r,j_{r,1}}) & \beta_r(t_{r,j_{r,1}}^\epsilon) & \cdots & \beta_r(t_{r,j_{r,k_r}}) & \beta_r(t_{r,j_{r,k_r}}^\epsilon) & \beta_r(M) & \beta_r(2M) & \cdots & \beta_r((d+r-1-2k)M) \end{vmatrix}.$$

²For example, for $d = 8$, $r = 3$, $k = 4$, $|U_1| = |U_3| = 1$, $|U_2| = 2$, and $\nu_i = 3 - i$, $i = 1, 2, 3$, $H_U(\mathbf{x})$ is the 11×11 determinant:

$$- \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ x_3 & \alpha_{1,j_{1,1}} \tau^2 & \alpha_{1,j_{1,1}}^\epsilon \tau^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_4 & 0 & 0 & \alpha_{2,j_{2,1}} \tau & \alpha_{2,j_{2,1}}^\epsilon \tau & \alpha_{2,j_{2,2}} \tau & \alpha_{2,j_{2,2}}^\epsilon \tau & 0 & 0 & 0 & 0 \\ x_5 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_{3,j_{3,1}} & \alpha_{3,j_{3,1}}^\epsilon & M & 2M \\ x_6 & \alpha_{1,j_{1,1}}^2 \tau^4 & (\alpha_{1,j_{1,1}}^\epsilon)^2 \tau^4 & \alpha_{2,j_{2,1}}^2 \tau^2 & (\alpha_{2,j_{2,1}}^\epsilon)^2 \tau^2 & \alpha_{2,j_{2,2}}^2 \tau^2 & (\alpha_{2,j_{2,2}}^\epsilon)^2 \tau^2 & \alpha_{3,j_{3,1}}^2 & (\alpha_{3,j_{3,1}}^\epsilon)^2 & M^2 & (2M)^2 \\ x_7 & \alpha_{1,j_{1,1}}^3 \tau^6 & (\alpha_{1,j_{1,1}}^\epsilon)^3 \tau^6 & \alpha_{2,j_{2,1}}^3 \tau^3 & (\alpha_{2,j_{2,1}}^\epsilon)^3 \tau^3 & \alpha_{2,j_{2,2}}^3 \tau^3 & (\alpha_{2,j_{2,2}}^\epsilon)^3 \tau^3 & \alpha_{3,j_{3,1}}^3 & (\alpha_{3,j_{3,1}}^\epsilon)^3 & M^3 & (2M)^3 \\ x_8 & \alpha_{1,j_{1,1}}^4 \tau^8 & (\alpha_{1,j_{1,1}}^\epsilon)^4 \tau^8 & \alpha_{2,j_{2,1}}^4 \tau^4 & (\alpha_{2,j_{2,1}}^\epsilon)^4 \tau^4 & \alpha_{2,j_{2,2}}^4 \tau^4 & (\alpha_{2,j_{2,2}}^\epsilon)^4 \tau^4 & \alpha_{3,j_{3,1}}^4 & (\alpha_{3,j_{3,1}}^\epsilon)^4 & M^4 & (2M)^4 \\ x_9 & \alpha_{1,j_{1,1}}^5 \tau^{10} & (\alpha_{1,j_{1,1}}^\epsilon)^5 \tau^{10} & \alpha_{2,j_{2,1}}^5 \tau^5 & (\alpha_{2,j_{2,1}}^\epsilon)^5 \tau^5 & \alpha_{2,j_{2,2}}^5 \tau^5 & (\alpha_{2,j_{2,2}}^\epsilon)^5 \tau^5 & \alpha_{3,j_{3,1}}^5 & (\alpha_{3,j_{3,1}}^\epsilon)^5 & M^5 & (2M)^5 \\ x_{10} & \alpha_{1,j_{1,1}}^6 \tau^{12} & (\alpha_{1,j_{1,1}}^\epsilon)^6 \tau^{12} & \alpha_{2,j_{2,1}}^6 \tau^6 & (\alpha_{2,j_{2,1}}^\epsilon)^6 \tau^6 & \alpha_{2,j_{2,2}}^6 \tau^6 & (\alpha_{2,j_{2,2}}^\epsilon)^6 \tau^6 & \alpha_{3,j_{3,1}}^6 & (\alpha_{3,j_{3,1}}^\epsilon)^6 & M^6 & (2M)^6 \end{vmatrix}.$$

Observe now that we can transform $H_U(\mathbf{u})$ in the form of the determinant $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ of Lemma 2, where $n \leftarrow r$, $\kappa_i \leftarrow 2k_i + 1$, $\kappa_r \leftarrow 2k_r + d + r - 1 - 2k$, $\kappa_j \leftarrow 2k_j$ for $1 \leq j < r$ and $j \neq i$, and $\beta_j \leftarrow \nu_j$ for $1 \leq j \leq r$, by means of the following determinant transformations:

- (i) By subtracting rows 2 to r of $H_U(\mathbf{u})$ from its first row.
- (ii) By shifting the first column of $H_U(\mathbf{u})$ to the right via an even number of column swaps. More precisely, to transform $H_U(\mathbf{u})$ in the form of $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$, we need to shift the first column of $H_U(\mathbf{u})$ to the right so that the values $t_{i,\lambda}, t_{i,j_{i,1}}, t_{i,j_{i,1}}^\epsilon, t_{i,j_{i,2}}, t_{i,j_{i,2}}^\epsilon, \dots, t_{i,j_{i,k_i}}, t_{i,j_{i,k_i}}^\epsilon$ appear consecutively in the columns of $H_U(\mathbf{u})$ and in increasing order. To do that we always need an even number of column swaps, due to the way we have chosen ϵ . More precisely, we first need to shift the first column of $H_U(\mathbf{u})$ through the $2 \sum_{j=1}^{i-1} k_j$ columns to its right; this is, obviously, done via an even number of column swaps. Consider the following cases:
 - If $\lambda < j_{i,1}$, then we are done: $H_U(\mathbf{u})$ is in the desired form.
 - If $\lambda > j_{i,k_i}$, notice that due to the way we have chosen ϵ , we have $t_{i,\lambda} > t_{i,j_{i,k_i}}^\epsilon$; therefore, we need to perform another $2k_i$ column swaps in order to shift $(\beta_i(t_{i,\lambda}))$ to its final place, i.e., to the right of $(\beta_i(t_{i,j_{i,k_i}}^\epsilon))$. In other words, in this case $H_U(\mathbf{u})$ can be transformed to the form of $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ with a total of $2 \sum_{j=1}^i k_j$ column swaps.
 - Finally, if $j_{i,1} < \lambda < j_{i,k_i}$, there exists some ξ with $1 \leq \xi < k_i$, such that $j_{i,\xi} < \lambda < j_{i,\xi+1}$. Since $t_{i,\lambda} > t_{i,j_{i,\xi}}^\epsilon$, due to the way we have chosen ϵ , we need another 2ξ column swaps to place $(\beta_i(t_{i,\lambda}))$ to the right of $(\beta_i(t_{i,j_{i,\xi}}^\epsilon))$. Hence, in this case, $H_U(\mathbf{u})$ can be transformed to the form of $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ with a total of $2(\xi + \sum_{j=1}^{i-1} k_j)$ column swaps.

Applying now Lemma 2, we deduce that there exists a value τ_0 for τ , such that for all $\tau \in (0, \tau_0)$, the determinant $H_U(\mathbf{u})$ is strictly positive.

We thus conclude that, for any specific choice of U , and for any specific vertex $\mathbf{u} \in \mathcal{U} \setminus U$, there exists some $\tau_0 > 0$ (cf. Lemma 2) that depends on \mathbf{u} and U , such that for all $\tau \in (0, \tau_0)$, $H_U(\mathbf{u}) > 0$. Since for each k with $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$, the number of \mathcal{U} -spanning subsets U of size k of \mathcal{U} is $\Phi_k(n_1, n_2, \dots, n_r)$, while for each such subset U we need to consider the $(\sum_{i=1}^r n_i - k)$ vertices in $\mathcal{U} \setminus U$, it suffices to consider a positive value τ^* for τ that is small enough, so that all

$$\sum_{k=r}^{\lfloor \frac{d+r-1}{2} \rfloor} \left(\sum_{i=1}^r n_i - k \right) \Phi_k(n_1, n_2, \dots, n_r)$$

possible determinants $H_U(\mathbf{u})$ are strictly positive. For $\tau \leftarrow \tau^*$, our analysis above immediately implies that for each \mathcal{U} -spanning subset U of \mathcal{U} the equation $H_U(\mathbf{x}) = 0$, $\mathbf{x} \in \mathbb{R}^{d+r-1}$, is the equation of a supporting hyperplane for \mathcal{Q} passing through the vertices of U , and those only. In other words, every \mathcal{U} -spanning subset U of \mathcal{U} , where $|U| = k$ and $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$, defines a $(k-1)$ -face of \mathcal{Q} , which means that $f_{k-1}(\mathcal{W}_{\mathcal{Q}}) = \Phi_k(n_1, n_2, \dots, n_r)$, for all $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$. \square

We assume we have chosen τ to be equal to τ^* , and call \mathcal{U}_i^* , Q_i^* , $1 \leq i \leq r$, the corresponding vertex sets and $(d-r+1)$ -polytopes. Let $\mathcal{U}^* = \cup_{i=1}^r \mathcal{U}_i^*$, and call \mathcal{Q}^* the Cayley polytope of $Q_1^*, Q_2^*, \dots, Q_r^*$. We are going to perturb the vertex sets \mathcal{U}_i^* to get the vertex sets \mathcal{V}_i , $1 \leq i \leq r$, by considering vertices on the curves $\gamma_i(t; \zeta)$ with $\zeta > 0$, instead of the curves $\gamma_i(t)$. More precisely, define the sets \mathcal{V}_i , $1 \leq i \leq r$, as follows:

$$\mathcal{V}_i = \{\gamma_i(\alpha_{i,1}(\tau^*)^{\nu_i}; \zeta), \gamma_i(\alpha_{i,2}(\tau^*)^{\nu_i}; \zeta), \dots, \gamma_i(\alpha_{i,n_i}(\tau^*)^{\nu_i}; \zeta)\},$$

where $\zeta > 0$. Let $\mathcal{V} = \cup_{i=1}^r \mathcal{V}_i$, call \mathbf{V} the partition of \mathcal{V} into its subsets $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r$, and let P_i be the d -polytope whose vertex set is \mathcal{V}_i . It is easy to verify that:

Claim 4. *For any $\zeta > 0$, P_i is a neighborly d -polytope.*

Proof. Recall that the vertices of P_i are taken from the moment-like curve:

$$\gamma_i(t; \zeta) = (\overbrace{\zeta t^{d-r+2}, \zeta t^{d-r+3}, \dots, \zeta t^{d-r+i}}^{i-1}, t, \overbrace{\zeta t^{d-r+i+1}, \dots, \zeta t^{d-1}}^{r-i}, \overbrace{\zeta t^d, t^2, \dots, t^{d-r+1}}^{d-r}). \quad (9)$$

Let $t_j = \alpha_{i,j}(\tau^*)^{\nu_i}$, $t_j^\epsilon = t_j + \epsilon$, $1 \leq j \leq n_i$, and $M > t_{n_i}^\epsilon$, where ϵ is a small positive constant chosen so that $t_j^\epsilon < t_{j+1}$, for all $1 \leq j < n_i$.

We will first show that P_i is d -dimensional. Consider a subset V of \mathcal{V}_i of size $d+1$, and let $\gamma_i(t_{j_1}; \zeta), \gamma_i(t_{j_2}; \zeta), \dots, \gamma_i(t_{j_{d+1}}; \zeta)$, be the vertices in V , where $j_1 < j_2 < \dots < j_{d+1}$. Define the $(d+1) \times (d+1)$ determinant $F(\zeta)$, corresponding to the volume of $\text{conv}(V)$ in \mathbb{R}^d :

$$F(\zeta) = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ \gamma_i(t_{j_1}; \zeta) & \gamma_i(t_{j_2}; \zeta) & \gamma_i(t_{j_3}; \zeta) & \dots & \gamma_i(t_{j_{d+1}}; \zeta) \end{vmatrix}.$$

It is easy to verify that $F(\zeta) = (-1)^{i-1+(r-1)(d-r)} \zeta^{r-1} \text{VD}(\mathbf{t})$, where $\mathbf{t} = (t_{j_1}, t_{j_2}, \dots, t_{j_{d+1}})$; recall that $\text{VD}(\mathbf{x})$ denotes the Vandermonde determinant corresponding to the vector \mathbf{x} (cf. (13)). Since the elements in \mathbf{t} are in strictly increasing order, we immediately conclude that $\text{VD}(\mathbf{t}) > 0$. This further implies that $F(\zeta) \neq 0$, for any $\zeta > 0$. Hence, the polytope P_i is d -dimensional, since it contains at least one d -dimensional simplex.

We will now show that P_i is neighborly. Consider a subset V of \mathcal{V}_i of size $k \leq \lfloor \frac{d}{2} \rfloor$, and let $\gamma_i(t_{j_1}; \zeta), \gamma_i(t_{j_2}; \zeta), \dots, \gamma_i(t_{j_k}; \zeta)$ be the vertices of P_i in V , where $j_1 < j_2 < \dots < j_k$. Define the $(d+1) \times (d+1)$ determinant $H_V(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^d$, as follows³:

$$H_V(\mathbf{x}) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & \dots & 1 \\ \mathbf{x} & \gamma_i(t_{j_1}) & \gamma_i(t_{j_1}^\epsilon) & \gamma_i(t_{j_2}) & \gamma_i(t_{j_2}^\epsilon) & \dots & \gamma_i(t_{j_k}) & \gamma_i(t_{j_k}^\epsilon) & \gamma_i(M) & \gamma_i(2M) & \dots & \gamma_i((d-2k)M) \end{vmatrix}.$$

Observe that

$$H_V(\gamma_i(t)) = (-1)^{i-1+(r-1)(d-r)} \zeta^{r-1} \text{VD}(\mathbf{T}),$$

where $\mathbf{T} = (t, t_{j_1}, t_{j_1}^\epsilon, t_{j_2}, t_{j_2}^\epsilon, \dots, t_{j_k}, t_{j_k}^\epsilon, M, 2M, \dots, (d-2k)M)$. In other words, for any $\zeta > 0$, $H_V(\gamma_i(t))$ is a polynomial in t of degree d , which has d distinct real roots, namely, $t_{j_1}, t_{j_2}, \dots, t_{j_k}, t_{j_1}^\epsilon, t_{j_2}^\epsilon, \dots, t_{j_k}^\epsilon$, and $M, 2M, \dots, (d-2k)M$. Observe, also, that there always exists an even number of roots⁴ of $H_V(\gamma_i(t))$ between $t = t_\mu$ and $t = t_\xi$ for any μ, ξ with $1 \leq \mu \neq \xi \leq n_i$ and $\mu, \xi \notin \{j_1, j_2, \dots, j_k\}$. This immediately implies that $H_V(\gamma_i(t))$ has always the same sign for any t_ℓ with $\ell \notin \{j_1, j_2, \dots, j_k\}$, which further implies that $H_V(\mathbf{v})$ has the same sign for any $\mathbf{v} \in \mathcal{V}_i \setminus V$. In other words, $H_V(\mathbf{x}) = 0$ is the equation of a supporting hyperplane of P_i , passing through the vertices of V , and those only.

Since we have chosen V arbitrarily, the same holds for any $V \subseteq \mathcal{V}_i$ with $|V| = k \leq \lfloor \frac{d}{2} \rfloor$. Thus, for any subset V of \mathcal{V}_i of size $k \leq \lfloor \frac{d}{2} \rfloor$, V defines a $(k-1)$ -face of P_i , i.e., P_i is neighborly. \square

Let $\mathcal{P} = \text{conv}(\mathcal{C}(\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_r))$ be the Cayley polytope of P_1, P_2, \dots, P_r , and let $\mathcal{W}_{\mathcal{P}}$ be the set of faces of \mathcal{P} with non-empty intersection with the d -flat \overline{W} of $\mathbb{R}^{r-1} \times \mathbb{R}^d$ (cf. (6)), i.e., $\mathcal{W}_{\mathcal{P}}$ consists of all the faces of \mathcal{P} , the vertex set of which is a \mathbf{V}' -spanning subset of \mathcal{V}' , where $\mathbf{V}' = \{\mathcal{V}'_1, \mathcal{V}'_2, \dots, \mathcal{V}'_r\}$, $\mathcal{V}' = \cup_{i=1}^r \mathcal{V}'_i$, and $\mathcal{V}'_i = \mu_i(\mathcal{V}_i)$, for $1 \leq i \leq r$.

The following lemma establishes the second, and final, step of our construction.

³For d even and $k = \lfloor \frac{d}{2} \rfloor$ the columns of $H_V(\mathbf{x})$ corresponding to $M, 2M, \dots, (d-2k)M$ do not exist.

⁴For d even and $k = \lfloor \frac{d}{2} \rfloor$ the d real roots of $H_V(\gamma_i(t))$ are $t_{j_1}, t_{j_2}, \dots, t_{j_k}, t_{j_1}^\epsilon, t_{j_2}^\epsilon, \dots, t_{j_k}^\epsilon$.

Lemma 5. *There exists a sufficiently small positive value ζ^\diamond for ζ , such that*

$$f_{k-1}(\mathcal{W}_P) = \Phi_k(n_1, n_2, \dots, n_r), \quad r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor.$$

Proof. As in the proof of Lemma 3, and in order to simplify the notation used in the proof, we identify \mathcal{V}'_i , \mathcal{V}' and \mathcal{V}' with their pre-images under the Cayley embedding, namely, \mathcal{V}_i , \mathcal{V} and \mathcal{V} , respectively.

Similarly to what we have done in the proof of Lemma 3, let $\alpha_{i,j}^\epsilon = \alpha_{i,j} + \epsilon$, $t_{i,j} = \alpha_{i,j}(\tau^\star)^{\nu_i}$, $t_{i,j}^\epsilon = \alpha_{i,j}^\epsilon(\tau^\star)^{\nu_i}$, $1 \leq j \leq n_i$, $1 \leq i \leq r$, and M be a positive real number such that $M > \alpha_{r,n_r}^\epsilon = t_{r,n_r}^\epsilon$, where $\epsilon > 0$ is chosen such that $\alpha_{i,j}^\epsilon < \alpha_{i,j+1}$, for all $1 \leq j < n_i$, and for all $1 \leq i \leq r$.

Choose a \mathcal{V} -spanning subset V of \mathcal{V} of size k , and denote by k_i the cardinality of $V_i = V \cap \mathcal{V}_i$, $1 \leq i \leq r$. Considering ζ as a small positive parameter, let $\beta_i(t_{i,j_{i,1}}; \zeta), \beta_i(t_{i,j_{i,2}}; \zeta), \dots, \beta_i(t_{i,j_{i,k_i}}; \zeta)$ be the vertices in V_i , where $j_{i,1} < j_{i,2} < \dots < j_{i,k_i}$. Let $\mathbf{x} = (x_1, x_2, \dots, x_{d+r-1})$ and define the $(d+r) \times (d+r)$ determinant $F_V(\mathbf{x}; \zeta)$ as⁵:

$$F_V(\mathbf{x}; \zeta) = (-1)^R \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ \mathbf{x} & \beta_1(t_{1,j_{1,1}}; \zeta) & \beta_1(t_{1,j_{1,1}}^\epsilon; \zeta) & \dots & \beta_1(t_{1,j_{1,k_1}}; \zeta) & \beta_1(t_{1,j_{1,k_1}}^\epsilon; \zeta) \\ & 1 & 1 & \dots & 1 & 1 \\ & \beta_2(t_{2,j_{2,1}}; \zeta) & \beta_2(t_{2,j_{2,1}}^\epsilon; \zeta) & \dots & \beta_2(t_{2,j_{2,k_2}}; \zeta) & \beta_2(t_{2,j_{2,k_2}}^\epsilon; \zeta) \\ \dots & & & & & \\ \dots & \beta_{r-1}(t_{r-1,j_{r-1,1}}; \zeta) & \beta_{r-1}(t_{r-1,j_{r-1,1}}^\epsilon; \zeta) & \dots & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}^\epsilon; \zeta) & \beta_{r-1}(t_{r-1,j_{r-1,k_{r-1}}}^\epsilon; \zeta) \\ & 1 & 1 & \dots & 1 & 1 \\ & \beta_r(t_{r,j_{r,1}}; \zeta) & \beta_r(t_{r,j_{r,1}}^\epsilon; \zeta) & \dots & \beta_r(t_{r,j_{r,k_r}}; \zeta) & \beta_r(t_{r,j_{r,k_r}}^\epsilon; \zeta) \\ & 1 & 1 & \dots & 1 & 1 \\ & \beta_r(M; \zeta) & \beta_r(2M; \zeta) & \dots & \beta_r((d+r-1-2k)M; \zeta) & \end{vmatrix}, \quad (10)$$

where $R = \frac{r(r-1)}{2}$. As for the determinant $H_U(\mathbf{x})$, we can alternatively describe $F_V(\mathbf{x}; \zeta)$ as follows:

- (i) The first column of $F_V(\mathbf{x}; \zeta)$ is $\begin{pmatrix} 1 \\ \mathbf{x} \end{pmatrix}$.
- (ii) For i ranging from 1 to r , and for λ ranging from 1 to k_i , the next k_i pairs of columns of $F_V(\mathbf{x})$ are $\begin{pmatrix} 1 \\ \beta_i(t_{i,j_{i,\lambda}}; \zeta) \end{pmatrix}$ and $\begin{pmatrix} 1 \\ \beta_i(t_{i,j_{i,\lambda}}^\epsilon; \zeta) \end{pmatrix}$.

⁵For example, for $d = 8$, $r = 3$, $k = 4$, $|V_1| = |V_3| = 1$, $|V_2| = 2$, and $\nu_i = 3 - i$, $i = 1, 2, 3$, $F_V(\mathbf{x}; \zeta)$ is the 11×11 determinant:

$$- \begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ x_2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ x_3 & \alpha_{1,j_{1,1}} \tau^2 & \alpha_{1,j_{1,1}}^\epsilon \tau^2 & \zeta \alpha_{2,j_{2,1}}^7 \tau^7 & \zeta (\alpha_{2,j_{2,1}}^\epsilon)^7 \tau^7 & \zeta \alpha_{2,j_{2,2}}^7 \tau^7 & \zeta (\alpha_{2,j_{2,2}}^\epsilon)^7 \tau^7 & \zeta \alpha_{3,j_{3,1}}^7 & \zeta (\alpha_{3,j_{3,1}}^\epsilon)^7 & \zeta M^7 & \zeta (2M)^7 \\ x_4 & \zeta \alpha_{1,j_{1,1}}^7 \tau^{14} & \zeta (\alpha_{1,j_{1,1}}^\epsilon)^7 \tau^{14} & \alpha_{2,j_{2,1}} \tau & \alpha_{2,j_{2,1}}^\epsilon \tau & \alpha_{2,j_{2,2}} \tau & \alpha_{2,j_{2,2}}^\epsilon \tau & \zeta \alpha_{3,j_{3,1}}^8 & \zeta (\alpha_{3,j_{3,1}}^\epsilon)^8 & \zeta M^8 & \zeta (2M)^8 \\ x_5 & \zeta \alpha_{1,j_{1,1}}^8 \tau^{16} & \zeta (\alpha_{1,j_{1,1}}^\epsilon)^8 \tau^{16} & \zeta \alpha_{2,j_{2,1}}^8 \tau^8 & \zeta (\alpha_{2,j_{2,1}}^\epsilon)^8 \tau^8 & \zeta \alpha_{2,j_{2,2}}^8 \tau^8 & \zeta (\alpha_{2,j_{2,2}}^\epsilon)^8 \tau^8 & \alpha_{3,j_{3,1}} & \alpha_{3,j_{3,1}}^\epsilon & M & 2M \\ x_6 & \alpha_{1,j_{1,1}}^2 \tau^4 & (\alpha_{1,j_{1,1}}^\epsilon)^2 \tau^4 & \alpha_{2,j_{2,1}}^2 \tau^2 & (\alpha_{2,j_{2,1}}^\epsilon)^2 \tau^2 & \alpha_{2,j_{2,2}}^2 \tau^2 & (\alpha_{2,j_{2,2}}^\epsilon)^2 \tau^2 & \alpha_{3,j_{3,1}}^2 & (\alpha_{3,j_{3,1}}^\epsilon)^2 & M^2 & (2M)^2 \\ x_7 & \alpha_{1,j_{1,1}}^3 \tau^6 & (\alpha_{1,j_{1,1}}^\epsilon)^3 \tau^6 & \alpha_{2,j_{2,1}}^3 \tau^3 & (\alpha_{2,j_{2,1}}^\epsilon)^3 \tau^3 & \alpha_{2,j_{2,2}}^3 \tau^3 & (\alpha_{2,j_{2,2}}^\epsilon)^3 \tau^3 & \alpha_{3,j_{3,1}}^3 & (\alpha_{3,j_{3,1}}^\epsilon)^3 & M^3 & (2M)^3 \\ x_8 & \alpha_{1,j_{1,1}}^4 \tau^8 & (\alpha_{1,j_{1,1}}^\epsilon)^4 \tau^8 & \alpha_{2,j_{2,1}}^4 \tau^4 & (\alpha_{2,j_{2,1}}^\epsilon)^4 \tau^4 & \alpha_{2,j_{2,2}}^4 \tau^4 & (\alpha_{2,j_{2,2}}^\epsilon)^4 \tau^4 & \alpha_{3,j_{3,1}}^4 & (\alpha_{3,j_{3,1}}^\epsilon)^4 & M^4 & (2M)^4 \\ x_9 & \alpha_{1,j_{1,1}}^5 \tau^{10} & (\alpha_{1,j_{1,1}}^\epsilon)^5 \tau^{10} & \alpha_{2,j_{2,1}}^5 \tau^5 & (\alpha_{2,j_{2,1}}^\epsilon)^5 \tau^5 & \alpha_{2,j_{2,2}}^5 \tau^5 & (\alpha_{2,j_{2,2}}^\epsilon)^5 \tau^5 & \alpha_{3,j_{3,1}}^5 & (\alpha_{3,j_{3,1}}^\epsilon)^5 & M^5 & (2M)^5 \\ x_{10} & \alpha_{1,j_{1,1}}^6 \tau^{12} & (\alpha_{1,j_{1,1}}^\epsilon)^6 \tau^{12} & \alpha_{2,j_{2,1}}^6 \tau^6 & (\alpha_{2,j_{2,1}}^\epsilon)^6 \tau^6 & \alpha_{2,j_{2,2}}^6 \tau^6 & (\alpha_{2,j_{2,2}}^\epsilon)^6 \tau^6 & \alpha_{3,j_{3,1}}^6 & (\alpha_{3,j_{3,1}}^\epsilon)^6 & M^6 & (2M)^6 \end{vmatrix}.$$

- (iii) For λ ranging from 1 to $d+r-1-2k$, the last $d+r-1-2k$ columns of $F_V(\mathbf{x}; \zeta)$ are $(\beta_r(\lambda M; \zeta))^1$. Notice that if $k = \lfloor \frac{d+r-1}{2} \rfloor$ and $d+r-1$ is even, this last category of columns of $F_V(\mathbf{x}; \zeta)$ does not exist.

The equation $F_V(\mathbf{x}; \zeta) = 0$ is the equation of a hyperplane in \mathbb{R}^{d+r-1} that passes through the points in V . We are going to show that for all vertices $\mathbf{v} \in \mathcal{V} \setminus V$, we have $F_V(\mathbf{v}; \zeta) > 0$ for sufficiently small ζ .

Indeed, choose some $\mathbf{v} \in \mathcal{V} \setminus V$, and suppose that $\mathbf{v} \in \mathcal{V}_i \setminus V$. Then \mathbf{v} is of the form $\mathbf{v} = \beta_i(t_{i,\lambda}; \zeta)$, $\zeta > 0$, for some $\lambda \notin \{j_{i,1}, j_{i,2}, \dots, j_{i,k_i}\}$. Let $\mathbf{u}^* = \beta_i(t_{i,\lambda}) = \beta_i(t_{i,\lambda}; 0)$. In more geometric terms, we define \mathbf{u}^* to be the projection of \mathbf{v} on the d -flat $\mathbb{R}^{r-1} \times F_i$ of $\mathbb{R}^{r-1} \times \mathbb{R}^d$, or, equivalently, \mathbf{u}^* is the (unperturbed) vertex in $\mathcal{U}^* \setminus U^*$ that corresponds to \mathbf{v} , where U^* stands for the set of (unperturbed) vertices of \mathcal{U}^* that correspond to the vertices in V . Observe that $F_V(\mathbf{v}; \zeta)$ is a polynomial function in ζ , and thus it is continuous with respect to ζ for any $\zeta \in \mathbb{R}$. This implies that

$$\lim_{\zeta \rightarrow 0^+} F_V(\mathbf{v}; \zeta) = F_{U^*}(\mathbf{u}^*; 0) = H_{U^*}(\mathbf{u}^*), \quad (11)$$

where we used the fact that $\lim_{\zeta \rightarrow 0^+} \mathbf{v} = \mathbf{u}^*$, and observed that $F_{U^*}(\mathbf{u}^*; 0) = H_{U^*}(\mathbf{u}^*)$, where $H_{U^*}(\mathbf{x})$ is the determinant in relation (8) in the proof of Lemma 3, for $\tau \leftarrow \tau^*$. Since $H_{U^*}(\mathbf{u}^*) > 0$ (recall that we have chosen τ to be equal to τ^*), we conclude, from (11), that there exists some $\zeta_0 > 0$ that depends on \mathbf{v} and V , such that for all $\zeta \in (0, \zeta_0)$, $F_V(\mathbf{v}; \zeta) > 0$.

Since for each k with $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$, the number of \mathbf{V} -spanning subsets V of \mathcal{V} of size k is $\Phi_k(n_1, n_2, \dots, n_r)$, while for each such subset V we need to consider the $(\sum_{i=1}^r n_i - k)$ vertices in $\mathcal{V} \setminus V$, it suffices to consider a positive value ζ^\diamond for ζ that is small enough, so that all

$$\sum_{k=r}^{\lfloor \frac{d+r-1}{2} \rfloor} \left(\sum_{i=1}^r n_i - k \right) \Phi_k(n_1, n_2, \dots, n_r)$$

possible determinants $F_V(\mathbf{v}; \zeta)$ are strictly positive. Hence, for $\zeta \leftarrow \zeta^\diamond$, we have that for each \mathbf{V} -spanning subset V of \mathcal{V} the equation $F_V(\mathbf{x}; \zeta^\diamond) = 0$, $\mathbf{x} \in \mathbb{R}^{d+r-1}$, is the equation of a supporting hyperplane for \mathcal{P} passing through the vertices of V , and those only. In other words, every \mathbf{V} -spanning subset V of \mathcal{V} , where $|V| = k$ and $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$, defines a $(k-1)$ -face of \mathcal{P} , which means that $f_{k-1}(\mathcal{W}_{\mathcal{P}}) = \Phi_k(n_1, n_2, \dots, n_r)$, for all $r \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor$. \square

From Lemma 5, in conjunction with (5), we immediately arrive at the following theorem, which states the main result of this paper.

Theorem 6. *Let $d \geq 3$ and $2 \leq r \leq d-1$. There exist r neighborly d -polytopes P_1, P_2, \dots, P_r in \mathbb{R}^d , with n_1, n_2, \dots, n_r vertices, respectively, such that, for all $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor - r$:*

$$f_k(P_1 + P_2 + \dots + P_r) = \Phi_{k+r}(n_1, n_2, \dots, n_r) = \sum_{\substack{1 \leq s_i \leq n_i \\ s_1 + \dots + s_r = k+r}} \prod_{i=1}^r \binom{n_i}{s_i}.$$

4 Discussion

In this paper we have considered the problem of evaluating the maximum number of k -faces of the Minkowski sum of r d -polytopes in \mathbb{R}^d . We have shown, with the aid of the Cayley trick for Minkowski sums, that the trivial upper bound proved by Fukuda and Weibel [4, Lemma 8] is

attainable for any $d \geq 3$, $2 \leq r \leq d - 1$ and $0 \leq k \leq \lfloor \frac{d+r-1}{2} \rfloor - r$, which is a significant and non-trivial extension of their previous tightness result (cf. [4, Theorem 4]). A direct corollary of our lower bounds is that the complexity of the Minkowski sum of r n -vertex d -polytopes is in $\Theta(n^{\lfloor \frac{d+r-1}{2} \rfloor})$, for any fixed $d \geq 3$ and $2 \leq r \leq d - 1$. We conjecture that the lower bound construction presented in this paper, gives, in fact, the maximum possible number of k -faces for the Minkowski sum of r d -polytopes for any $d \geq 3$, $2 \leq r \leq d - 1$, and for all $0 \leq k \leq d - 1$. Our conjecture has been positively asserted for the case of two d -polytopes (cf. [11, 12]).

Given the results in this paper, as well as the tight bounds in [11, 12] and [16], the obvious remaining open problem is to devise a tight expression for the maximum number of k -faces of the Minkowski sum of r d -polytopes for: (i) $d \geq 4$, $3 \leq r \leq d - 1$ and $\lfloor \frac{d+r-1}{2} \rfloor - r < k \leq d - 1$, and (ii) for $r \geq d \geq 3$ and $1 \leq k \leq d - 1$. Another relevant open problem is to express the maximum number of k -faces of the Minkowski sum of r d -polytopes as a function of the number of facets of the polytopes. Results in this direction are known for 2- and 3-polytopes only (cf. [15], [2]).

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A Proof of Lemma 2

We start by introducing what is known as *Laplace's Expansion Theorem* for determinants (see [5, 8] for details and proofs). Consider a $n \times n$ matrix A . Let $\mathbf{r} = (r_1, r_2, \dots, r_k)$, be a vector of k row indices for A , where $1 \leq k < n$ and $1 \leq r_1 < r_2 < \dots < r_k \leq n$. Let $\mathbf{c} = (c_1, c_2, \dots, c_k)$ be a vector of k column indices for A , where $1 \leq k < n$ and $1 \leq c_1 < c_2 < \dots < c_k \leq n$. We denote by $S(A; \mathbf{r}, \mathbf{c})$ the $k \times k$ submatrix of A constructed by keeping the entries of A that belong to a row in \mathbf{r} and a column in \mathbf{c} . The *complementary submatrix* for $S(A; \mathbf{r}, \mathbf{c})$, denoted by $\bar{S}(A; \mathbf{r}, \mathbf{c})$, is the $(n - k) \times (n - k)$ submatrix of A constructed by removing the rows and columns of A in \mathbf{r} and \mathbf{c} , respectively. Then, the determinant of A can be computed by expanding in terms of the k columns of A in \mathbf{c} according to the following theorem.

Theorem 7 (Laplace's Expansion Theorem). *Let A be a $n \times n$ matrix. Let $\mathbf{c} = (c_1, c_2, \dots, c_k)$ be a vector of k column indices for A , where $1 \leq k < n$ and $1 \leq c_1 < c_2 < \dots < c_k \leq n$. Then:*

$$\det(A) = \sum_{\mathbf{r}} (-1)^{|\mathbf{r}|+|\mathbf{c}|} \det(S(A; \mathbf{r}, \mathbf{c})) \det(\bar{S}(A; \mathbf{r}, \mathbf{c})), \quad (12)$$

where $|\mathbf{r}| = r_1 + r_2 + \dots + r_k$, $|\mathbf{c}| = c_1 + c_2 + \dots + c_k$, and the summation is taken over all row vectors $\mathbf{r} = (r_1, r_2, \dots, r_k)$ of k row indices for A , where $1 \leq r_1 < r_2 < \dots < r_k \leq n$.

Given a vector of $n \geq 2$ real numbers $\mathbf{x} = (x_1, x_2, \dots, x_n)$, the *Vandermonde determinant* $\text{VD}(\mathbf{x})$ of \mathbf{x} is the $n \times n$ determinant

$$\text{VD}(\mathbf{x}) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_n \\ x_1^2 & x_2^2 & \dots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1 \leq i < j \leq n} (x_j - x_i) \quad (13)$$

From the above expression, it is readily seen that if the elements of \mathbf{x} are in strictly increasing order, then $\text{VD}(\mathbf{x}) > 0$. A generalization of the Vandermonde determinant is the generalized Vandermonde determinant: if, in addition to \mathbf{x} , we specify a vector of exponents $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_n)$, where we require that $0 \leq \mu_1 < \mu_2 < \dots < \mu_n$, we can define the *generalized Vandermonde determinant* $\text{GVD}(\mathbf{x}; \boldsymbol{\mu})$ as the $n \times n$ determinant:

$$\text{GVD}(\mathbf{x}; \boldsymbol{\mu}) = \begin{vmatrix} x_1^{\mu_1} & x_2^{\mu_1} & \dots & x_n^{\mu_1} \\ x_1^{\mu_2} & x_2^{\mu_2} & \dots & x_n^{\mu_2} \\ x_1^{\mu_3} & x_2^{\mu_3} & \dots & x_n^{\mu_3} \\ \vdots & \vdots & & \vdots \\ x_1^{\mu_n} & x_2^{\mu_n} & \dots & x_n^{\mu_n} \end{vmatrix}.$$

It is a well-known fact (e.g., see [6]) that, if the elements of \mathbf{x} are in strictly increasing order, then $\text{GVD}(\mathbf{x}; \boldsymbol{\mu}) > 0$.

We now restate Lemma 2 and prove it.

Lemma 2. *Let $\kappa_1, \dots, \kappa_n$ be $n \geq 2$ integers such that $\kappa_i \geq 2$, $1 \leq i \leq n$, and let $K = \sum_{i=1}^n \kappa_i$. Let $x_{i,j}$ be real numbers such that $0 < x_{i,1} < x_{i,2} < \dots < x_{i,\kappa_i}$, $1 \leq i \leq n$. Let β_i , $1 \leq i \leq n$ be non-negative integers such that $\beta_1 > \beta_2 > \dots > \beta_n \geq 0$. Finally, let τ be a positive real parameter, and define $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ to be the determinant:*

$$(-1)^N \begin{vmatrix} \overbrace{\begin{matrix} 1 & \dots & 1 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{matrix}}^{\kappa_1} & \overbrace{\begin{matrix} 0 & \dots & 0 \\ 1 & \dots & 1 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{matrix}}^{\kappa_2} & \overbrace{\begin{matrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 1 & \dots & 1 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{matrix}}^{\kappa_3} & \dots & \overbrace{\begin{matrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{matrix}}^{\kappa_n} \\ x_{1,1}\tau^{\beta_1} \dots x_{1,\kappa_1}\tau^{\beta_1} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & x_{2,1}\tau^{\beta_2} \dots x_{2,\kappa_2}\tau^{\beta_2} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & x_{3,1}\tau^{\beta_3} \dots x_{3,\kappa_3}\tau^{\beta_3} & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & \dots & 0 & \dots & x_{n,1}\tau^{\beta_n} \dots x_{n,\kappa_n}\tau^{\beta_n} \\ x_{1,1}^2\tau^{2\beta_1} \dots x_{1,\kappa_1}^2\tau^{2\beta_1} & x_{2,1}^2\tau^{2\beta_2} \dots x_{2,\kappa_2}^2\tau^{2\beta_2} & x_{3,1}^2\tau^{2\beta_3} \dots x_{3,\kappa_3}^2\tau^{2\beta_3} & \dots & x_{n,1}^2\tau^{2\beta_n} \dots x_{n,\kappa_n}^2\tau^{2\beta_n} \\ x_{1,1}^3\tau^{3\beta_1} \dots x_{1,\kappa_1}^3\tau^{3\beta_1} & x_{2,1}^3\tau^{3\beta_2} \dots x_{2,\kappa_2}^3\tau^{3\beta_2} & x_{3,1}^3\tau^{3\beta_3} \dots x_{3,\kappa_3}^3\tau^{3\beta_3} & \dots & x_{n,1}^3\tau^{3\beta_n} \dots x_{n,\kappa_n}^3\tau^{3\beta_n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ x_{1,1}^m\tau^{m\beta_1} \dots x_{1,\kappa_1}^m\tau^{m\beta_1} & x_{2,1}^m\tau^{m\beta_2} \dots x_{2,\kappa_2}^m\tau^{m\beta_2} & x_{3,1}^m\tau^{m\beta_3} \dots x_{3,\kappa_3}^m\tau^{m\beta_3} & \dots & x_{n,1}^m\tau^{m\beta_n} \dots x_{n,\kappa_n}^m\tau^{m\beta_n} \end{vmatrix} \begin{matrix} \left. \vphantom{\begin{matrix} 1 & \dots & 1 \\ 0 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{matrix}} \right\} n \\ \left. \vphantom{\begin{matrix} 0 & x_{2,1}\tau^{\beta_2} \dots x_{2,\kappa_2}\tau^{\beta_2} \\ 0 & 0 & x_{3,1}\tau^{\beta_3} \dots x_{3,\kappa_3}\tau^{\beta_3} \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{matrix}} \right\} n \\ \left. \vphantom{\begin{matrix} x_{1,1}^2\tau^{2\beta_1} \dots x_{1,\kappa_1}^2\tau^{2\beta_1} \\ x_{1,1}^3\tau^{3\beta_1} \dots x_{1,\kappa_1}^3\tau^{3\beta_1} \\ \vdots \\ x_{1,1}^m\tau^{m\beta_1} \dots x_{1,\kappa_1}^m\tau^{m\beta_1} \end{matrix}} \right\} K - 2n \end{matrix}$$

where $N = \frac{n(n-1)}{2}$ and $m = K - 2n + 1$. Then, there exists some $\tau_0 > 0$, such that for all $\tau \in (0, \tau_0)$, the determinant $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ is strictly positive.

Proof. We denote by $\boldsymbol{\Delta}_{(\kappa_1, \dots, \kappa_n)}(\tau)$ the matrix corresponding to the determinant $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$. We are going to apply Laplace's Expansion Theorem to evaluate $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ in terms of the first κ_1 columns of $\boldsymbol{\Delta}_{(\kappa_1, \dots, \kappa_n)}(\tau)$, then the next κ_2 columns of $\boldsymbol{\Delta}_{(\kappa_1, \dots, \kappa_n)}(\tau)$, then the next κ_3 columns of $\boldsymbol{\Delta}_{(\kappa_1, \dots, \kappa_n)}(\tau)$, and so on. Let $K_i = \sum_{j=1}^i \kappa_j$, $1 \leq i \leq n$, and let \mathbf{c}_i , $1 \leq i \leq n$, be the column vector corresponding to the columns $K_{i-1} + 1$ to K_i (by convention $K_0 = 0$), i.e., $\mathbf{c}_i = (K_{i-1} + 1, K_{i-1} + 2, \dots, K_i - 1, K_i)$, $1 \leq i \leq n$. By applying Laplace's Expansion Theorem we get:

$$\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau) = \sum_{(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n)} (-1)^{\sigma(\mathbf{r}, \mathbf{c}) + N} \prod_{i=1}^n \det(S(\boldsymbol{\Delta}_{(\kappa_1, \dots, \kappa_n)}(\tau); \mathbf{r}_i, \mathbf{c}_i)) \quad (14)$$

where $\sigma(\mathbf{r}, \mathbf{c})$ is an expression that depends on \mathbf{r} and \mathbf{c} , while $\det(S(\boldsymbol{\Delta}_{(\kappa_1, \dots, \kappa_n)}(\tau); \mathbf{r}_i, \mathbf{c}_i))$ is the $\kappa_i \times \kappa_i$ submatrix of $\boldsymbol{\Delta}_{(\kappa_1, \dots, \kappa_n)}(\tau)$ formed by taking the elements of $\boldsymbol{\Delta}_{(\kappa_1, \dots, \kappa_n)}(\tau)$ that belong to the κ_i columns in \mathbf{c}_i and the κ_i rows in \mathbf{r}_i .

It is easy to verify that the above sum consists of $\prod_{i=1}^n \binom{K-K_{i-1}}{\kappa_i}$ terms. Observe that, among these terms:

- (i) all terms for which \mathbf{r}_i contains the j -th row, where $1 \leq j \leq 2n$ and $j \notin \{i, n+i\}$ vanish, since the matrix $S(\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau); \mathbf{r}_i, \mathbf{c}_i)$, consists of at least one row of zeros, and,
- (ii) all terms for which \mathbf{r}_i does not contain the i -th or the $(n+i)$ -th row vanish, since at least one of the matrices $S(\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau); \mathbf{r}_j, \mathbf{c}_j)$, $j \neq i$, consists of a row of zeros.

In other words, the row vector \mathbf{r}_i has to be of the form $\mathbf{r}_i = (i, n+i, r_{i,3}, r_{i,4}, \dots, r_{i,\kappa_i})$, while the remaining terms of the expansion are the $\prod_{i=1}^n \binom{K-2n-K_{i-1}}{\kappa_i-2}$ terms for which $\mathbf{r}_i = (i, n+i, r_{i,3}, r_{i,4}, \dots, r_{i,\kappa_i})$, with $2n+1 \leq r_{i,3} < r_{i,4} < \dots < r_{i,\kappa_i} \leq K$. For any given such \mathbf{r}_i , we have that $\det(S(\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau); \mathbf{r}_i, \mathbf{c}_i))$ is the $\kappa_i \times \kappa_i$ generalized Vandermonde determinant $\text{GVD}(\tau^{\beta_i} \mathbf{x}_i; \mathbf{r}_i - \boldsymbol{\alpha}_i)$, where $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \dots, x_{i,\kappa_i})$ and $\boldsymbol{\alpha}_i = (i, n+i-1, 2n-1, 2n-1, \dots, 2n-1) = i\mathbf{e}_1 + (n+i-1)\mathbf{e}_2 + (2n-1)\sum_{j=3}^{\kappa_i} \mathbf{e}_j$; here \mathbf{e}_j stands for the vector whose elements vanish, except for the j -th element, which is equal to 1.

We can, thus, simplify the expansion in (14) to get:

$$\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau) = \sum_{\substack{(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n) \\ \mathbf{r}_\nu = (\nu, n+\nu, r_{\nu,3}, \dots, r_{\nu,\kappa_\nu}) \\ 2n+1 \leq r_{\nu,3} < r_{\nu,4} < \dots < r_{\nu,\kappa_\nu} \leq K}} (-1)^{\sigma(\mathbf{r}, \mathbf{c})+N} \prod_{i=1}^n \text{GVD}(\tau^{\beta_i} \mathbf{x}_i; \mathbf{r}_i - \boldsymbol{\alpha}_i) \quad (15)$$

Since

$$\text{GVD}(\tau^{\beta_i} \mathbf{x}_i; \mathbf{r}_i - \boldsymbol{\alpha}_i) = \tau^{\beta_i |\mathbf{r}_i| - \beta_i |\boldsymbol{\alpha}_i|} \text{GVD}(\mathbf{x}_i; \mathbf{r}_i - \boldsymbol{\alpha}_i),$$

the expansion for $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ can be further rewritten as:

$$\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau) = \sum_{\substack{(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n) \\ \mathbf{r}_\nu = (\nu, n+\nu, r_{\nu,3}, \dots, r_{\nu,\kappa_\nu}) \\ 2n+1 \leq r_{\nu,3} < r_{\nu,4} < \dots < r_{\nu,\kappa_\nu} \leq K}} (-1)^{\sigma(\mathbf{r}, \mathbf{c})+N} \tau^{\sum_{i=1}^n \beta_i (|\mathbf{r}_i| - |\boldsymbol{\alpha}_i|)} \prod_{i=1}^n \text{GVD}(\mathbf{x}_i; \mathbf{r}_i - \boldsymbol{\alpha}_i) \quad (16)$$

Let $\boldsymbol{\rho} = (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \dots, \boldsymbol{\rho}_n)$ be the row vector for which $\boldsymbol{\rho}_i = (i, n+i, 2n+K_{i-1}-2(i-1)+1, 2n+K_{i-1}-2(i-1)+2, \dots, 2n+K_i-2i) = (i, n+i, 2(n-i)+K_{i-1}+3, 2(n-i)+K_{i-1}+4, \dots, 2(n-i)+K_i)$, $1 \leq i \leq n$. We claim that the minimum exponent for τ in (16) is attained when $\mathbf{r}_i \equiv \boldsymbol{\rho}_i$, for all $1 \leq i \leq n$. Suppose, on the contrary, that the minimum exponent for τ is attained for the row vector $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n) \neq \boldsymbol{\rho}$. Let i_1 , $1 \leq i_1 \leq n$, be the minimum index i for which $\mathbf{r}_i \neq \boldsymbol{\rho}_i$, and let r_{i_1, ℓ_1} , $3 \leq \ell_1 \leq \kappa_{i_1}$, be the first element of \mathbf{r}_{i_1} that differs from the corresponding element of $\boldsymbol{\rho}_{i_1}$, i.e., $r_{i_1, j} = \rho_{i_1, j}$, $1 \leq j \leq \ell_1 - 1$, and $r_{i_1, \ell_1} \neq \rho_{i_1, \ell_1}$. Since both \mathbf{r} and $\boldsymbol{\rho}$ are row vectors, there exists another index i_2 with $1 \leq i_1 < i_2 \leq n$, such that \mathbf{r}_{i_2} contains ρ_{i_1, ℓ_1} , and let ℓ_2 , $3 \leq \ell_2 \leq \kappa_{i_2}$, be the index of ρ_{i_1, ℓ_1} in \mathbf{r}_{i_2} , i.e., $r_{i_2, \ell_2} = \rho_{i_1, \ell_1}$. Define the row vector $\bar{\mathbf{r}} = (\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2, \dots, \bar{\mathbf{r}}_n)$ as follows. Let $\bar{\mathbf{r}}_i = \mathbf{r}_i$, for all $i \neq i_1, i_2$, $\bar{\mathbf{r}}_{i_1}$ has the same elements as \mathbf{r}_{i_1} , except for its ℓ_1 -th element, which is replaced by ρ_{i_1, ℓ_1} (i.e., $\bar{r}_{i_1, \ell_1} = \rho_{i_1, \ell_1} < r_{i_1, \ell_1}$), whereas $\bar{\mathbf{r}}_{i_2}$ has the same elements as \mathbf{r}_{i_2} , except for its ℓ_2 -th element, which is replaced by r_{i_1, ℓ_1} (i.e., $\bar{r}_{i_2, \ell_2} = r_{i_1, \ell_1} > \rho_{i_1, \ell_1} = r_{i_2, \ell_2}$). This implies that $|\bar{\mathbf{r}}_i| = |\mathbf{r}_i|$, for all $i \neq i_1, i_2$, whereas

$$\begin{aligned} |\bar{\mathbf{r}}_{i_1}| &= |\mathbf{r}_{i_1}| - r_{i_1, \ell_1} + \rho_{i_1, \ell_1}, \\ |\bar{\mathbf{r}}_{i_2}| &= |\mathbf{r}_{i_2}| - r_{i_2, \ell_2} + r_{i_1, \ell_1} = |\mathbf{r}_{i_2}| - \rho_{i_1, \ell_1} + r_{i_1, \ell_1}. \end{aligned}$$

Hence we get:

$$\begin{aligned}
\sum_{i=1}^n \beta_i |\bar{\mathbf{r}}_i| &= \sum_{\substack{i=1 \\ i \neq i_1, i_2}}^n \beta_i |\bar{\mathbf{r}}_i| + \beta_{i_1} |\bar{\mathbf{r}}_{i_1}| + \beta_{i_2} |\bar{\mathbf{r}}_{i_2}| \\
&= \sum_{\substack{i=1 \\ i \neq i_1, i_2}}^n \beta_i |\mathbf{r}_i| + \beta_{i_1} (|\mathbf{r}_{i_1}| - r_{i_1, \ell_1} + \rho_{i_1, \ell_1}) + \beta_{i_2} (|\mathbf{r}_{i_2}| - \rho_{i_1, \ell_1} + r_{i_1, \ell_1}) \\
&= \sum_{i=1}^n \beta_i |\mathbf{r}_i| + (\beta_{i_1} - \beta_{i_2})(\rho_{i_1, \ell_1} - r_{i_1, \ell_1}) \\
&< \sum_{i=1}^n \beta_i |\mathbf{r}_i|,
\end{aligned}$$

where we used that $\rho_{i_1, \ell_1} < r_{i_1, \ell_1}$ and $\beta_{i_1} > \beta_{i_2}$ (since $i_1 > i_2$). This, however, contradicts the minimality property of \mathbf{r} , which means that our assumption that $\mathbf{r} \neq \boldsymbol{\rho}$ is false.

For $\mathbf{r} \equiv \boldsymbol{\rho}$, it is easy to verify that

$$\sigma(\boldsymbol{\rho}, \mathbf{c}) = \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i} j + \sum_{i=1}^{n-1} \left[1 + (n+2-i) + \sum_{j=1}^{\kappa_i-2} [2(n+1-i) + j] \right]. \quad (17)$$

To see why this expression holds, consider expanding $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$ along the columns in \mathbf{c}_1 and the rows in $\boldsymbol{\rho}_1$. This contributes a term of $\sum_{j=1}^{\kappa_1} j$ to $\sigma(\boldsymbol{\rho}, \mathbf{c})$, corresponding to $|\mathbf{c}_1|$, and a term $1 + (n+1) + \sum_{j=1}^{\kappa_1-2} (2n+j)$ corresponding to $|\boldsymbol{\rho}_1|$. The remaining complementary $(K - \kappa_1) \times (K - \kappa_1)$ submatrix is then expanded along the column vector corresponding to its first κ_2 columns, and the row vector corresponding to its first row, its n -th row, as well as rows $2(n-1)+j$, with $1 \leq j \leq \kappa_2-2$. This contributes a term $\sum_{j=1}^{\kappa_2} j$ to $\sigma(\boldsymbol{\rho}, \mathbf{c})$ corresponding to its first κ_2 columns, and a term of $1 + n + \sum_{j=1}^{\kappa_2-2} (2(n-1) + j)$ corresponding to its rows. At the i -th step of this procedure, which is performed $(n-1)$ times, the remaining determinant corresponds to a $(K - K_{i-1}) \times (K - K_{i-1})$ submatrix of $\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)$, which is expanded with respect to its first κ_i columns, and with respect to its first and $(n+2-i)$ -th row, as well as its rows $2(n+1-i) + j$, where $1 \leq j \leq \kappa_i - 2$. Hence the contribution to $\sigma(\boldsymbol{\rho}, \mathbf{c})$ from the columns in \mathbf{c}_i is $\sum_{j=1}^{\kappa_i} j$, while the contribution from the rows in $\boldsymbol{\rho}_i$ is $1 + (n+2-i) + \sum_{j=1}^{\kappa_i-2} [2(n+1-i) + j]$. Expanding $\sigma(\boldsymbol{\rho}, \mathbf{c})$ from (17), and gathering terms appropriately, we get:

$$\begin{aligned}
\sigma(\boldsymbol{\rho}, \mathbf{c}) &= \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i} j + \sum_{i=1}^{n-1} \left[1 + (n+2-i) + \sum_{j=1}^{\kappa_i-2} [2(n+1-i) + j] \right] \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i} j + \sum_{i=1}^{n-1} (n+3-i) + 2 \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i-2} (n+1-i) + \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i-2} j \\
&= \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i} j + 3(n-1) + \sum_{i=1}^{n-1} (n-i) + 2 \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i-2} (n+1-i) + \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i} j - \sum_{i=1}^{n-1} [(\kappa_i - 1) + \kappa_i] \\
&= 2 \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i} j + 3(n-1) + \sum_{i=1}^{n-1} i + 2 \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i-2} (n+1-i) - 2 \sum_{i=1}^{n-1} \kappa_i + \sum_{i=1}^{n-1} 1
\end{aligned}$$

$$\begin{aligned}
&= 2 \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i} j + 4(n-1) + \frac{n(n-1)}{2} + 2 \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i-2} (n+1-i) - 2 \sum_{i=1}^{n-1} \kappa_i \\
&= 2 \left[\sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i} j + 2(n-1) + \sum_{i=1}^{n-1} \sum_{j=1}^{\kappa_i-2} (n+1-i) - \sum_{i=1}^{n-1} \kappa_i \right] + \frac{n(n-1)}{2}.
\end{aligned}$$

This means that $(-1)^{\sigma(\boldsymbol{\rho}, \mathbf{c})+N} = (-1)^{n(n-1)} = 1$, since $n(n-1)$ is always even. Therefore, relation (16) gives:

$$\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau) = \tau^{\sum_{i=1}^n \beta_i(|\boldsymbol{\rho}_i| - |\boldsymbol{\alpha}_i|)} \prod_{i=1}^n \text{GVD}(\mathbf{x}_i; \boldsymbol{\rho}_i - \boldsymbol{\alpha}_i) + O(\tau^{\sum_{i=1}^n \beta_i(|\boldsymbol{\rho}_i| - |\boldsymbol{\alpha}_i|) + 1}). \quad (18)$$

From relation (18) we immediately deduce that:

$$\lim_{\tau \rightarrow 0^+} \frac{\Delta_{(\kappa_1, \dots, \kappa_n)}(\tau)}{\tau^{\sum_{i=1}^n \beta_i(|\boldsymbol{\rho}_i| - |\boldsymbol{\alpha}_i|)}} = \prod_{i=1}^n \text{GVD}(\mathbf{x}_i; \boldsymbol{\rho}_i - \boldsymbol{\alpha}_i) \quad (19)$$

which establishes the claim of the lemma, since $\text{GVD}(\mathbf{x}_i; \boldsymbol{\rho}_i - \boldsymbol{\alpha}_i)$ is strictly positive, for all $1 \leq i \leq n$. \square